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NEW PHENOMENA IN THE BUCKLING OF ARCHES DESCRIBED BY REFINED THEORIES

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Abstract—In this paper we study the local and global behavior of buckled states of nonlinearly elastic circular arches under the action of hydrostatic pressure, concentrating on new effects exhibited by the solution branches. We employ very general (geometrically exact) models for arches that can suffer flexure, extension and shear. The constitutive equations used are intimately related to those of the three-dimensional theory of nonlinear elasticity. These new effects are caused by the interaction of the nonlinearity of material response, the nature of boundary conditions, the cross-sectional geometry, the thickness and the location of the material curve on which the pressure is applied.

1. INTRODUCTION

In the standard engineering theories of rods, both linear and nonlinear, the shape of a cross-section enters the constitutive equations only through its area and moments of inertia. In many such theories of elastic rods the bending couple vanishes where the change in curvature vanishes. If, however, we examine rod theories from the viewpoint of three-dimensional continuum mechanics, then we find that many of these common assumptions are unwarranted for large deformations. Instead, we confront a richer collection of constitutive functions with a detailed mathematical structure not suggested by that of any of the standard theories.

The purpose of this paper is to investigate the effect of these more refined models, inspired by the three-dimensional theory, on the local and global behavior of buckled states of nonlinearly elastic circular arches under the action of hydrostatic pressure, and to illustrate methods for treating some of the consequent technical difficulties. We encounter a variety of subtle new physical (and mathematical) phenomena, caused by the interaction of the nonlinearity of material response, the nature of boundary conditions, the cross-sectional geometry, the thickness, and the location of the material curve on which the pressure is applied. Indeed, the determination of the conditions under which there are unbuckled circular states (corresponding to trivial solutions) for arches with hinged ends requires the solution of an inverse problem that is strongly influenced by the constitutive functions.

The arches we study can suffer flexure, extension and shear. Our model also describes the plane-strain buckling of a sector of a cylindrical shell under hydrostatic pressure and the buckling of a current-carrying arch in the presence of a magnetic field acting perpendicular to the plane of the arch. Some of the phenomena we obtain have important implications for stability.

We formulate the governing geometrically exact theory of the planar deformation of nonlinearly elastic arches in Section 2. The behavior of solutions to the governing equations depends crucially on the constitutive hypotheses we impose. These are inspired by those for a general rod theory constructed by constraining the admissible displacements of a three-dimensional nonlinearly elastic body. This development, given in Section 3, has a character different from that of the rest of the paper. In Section 4 we give a very careful analysis of boundary conditions. In the rest of the paper we examine the consequences of the constitutive assumptions on the local and global behavior of branches of solutions of the governing nonlinear boundary-value problems.

Notation

Vectors (which are elements of Euclidean 3-space) are denoted by lower-case bold symbols. The dot product and cross product of (vectors) $\mathbf{u}$ and $\mathbf{v}$ are denoted by $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$. Partial derivatives are frequently denoted by subscripts. If f is a function of three variables x, y, z, then, e.g. $f(x,\cdot,z)$, denotes the resulting function of the second variable, when the first and third arguments are fixed at the values x and z.

2. FORMULATION OF THE GOVERNING EQUATIONS

In this section we formulate the equations governing the planar equilibria of nonlinearly elastic rods subject to hydrostatic pressure. It is then a trivial matter to specialize these equations to a circular arch.

Let $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ be a fixed right-handed, orthonormal basis for Euclidean 3-space $\mathbb{E}^3$. A *configuration* of a rod confined to deform in the $\{\mathbf{i},\mathbf{j}\}$-plane is described by a pair of continuously differentiable vector-valued functions

$$s\mapsto \mathbf{r}(s) \equiv x(s)\mathbf{i}+y(s)\mathbf{j},\quad \mathbf{b}(\theta(s)) \equiv -\sin\theta(s)\mathbf{i}+\cos\theta(s)\mathbf{j} \tag{1}$$

defined on an interval of the form $(-\alpha,\alpha)$. Here $\mathbf{r}$, which defines a curve in the $\{\mathbf{i},\mathbf{j}\}$-plane, is interpreted as some material curve in the rod regarded as a two-dimensional or three-dimensional body (e.g. the curve of centroids of the rod or a suitable curve lying on the boundary of the body). $\mathbf{b}(s)$ is interpreted as describing the orientation of the cross-section at s of the arch (see Fig. 1). We set

$$\mathbf{a}(\theta) \equiv -\mathbf{k}\times\mathbf{b}(\theta) \equiv \cos\theta(s)\mathbf{i}+\sin\theta(s)\mathbf{j}, \tag{2}$$

$$\mathbf{r}' = \nu\mathbf{a}+\eta\mathbf{b}. \tag{3}$$

Here and below the prime denotes ordinary differentiation, usually with respect to s. In the undeformed reference state we take

$$\nu = 1,\quad \eta = 0,\quad \theta(s) = \mathring{\theta}(s), \tag{4}$$

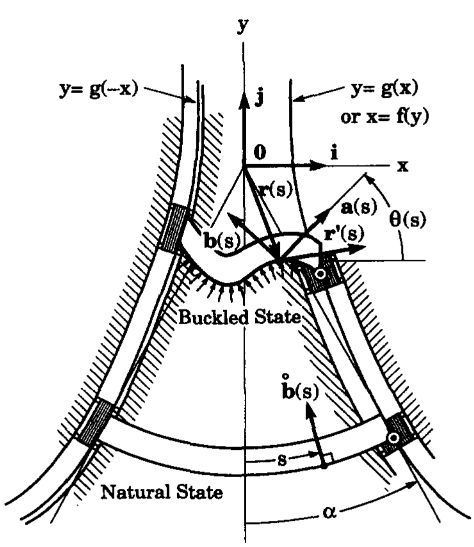

Fig. 1. Geometry of the natural reference configuration and of a typical buckled state of an arch subjected to hydrostatic pressure. The ends of the arch are confined to frictionless grooves with equations $y = g(\pm x)$. (The reference state of the arch is taken to be bowed down and the pressure p is taken to act upward in order to obtain sign conventions nicer than those for the traditional reversed situation.)

where $\mathring{\theta}$ is given. The *strains* for our problem are

$$(v, \eta, \mu \equiv \theta'). \tag{5}$$

As we shall show, a one-dimensional version of the requirement that the deformation preserve orientation locally is that there be a convex function $V(\cdot, s)$ on $\mathbb{R}$ with $V(0, s) = 0$ such that

$$v(s) > V(\theta'(s), s). \tag{6}$$

Let $\mathbf{n}(s) \equiv N(s)\mathbf{a}(s) + H(s)\mathbf{b}(s)$ and $M(s)\mathbf{k}$ be the resultant contact force and couple acting across section s in a deformed configuration. Other than reactions exerted at the ends of the rod, the only external force on it is a hydrostatic pressure of intensity p per unit deformed length of $\mathbf{r}$. The *equilibrium equations* are

$$\mathbf{n}' + p\mathbf{k} \times \mathbf{r}' = \mathbf{0}, \tag{7}$$

$$M' + \mathbf{k} \cdot (\mathbf{r}' \times \mathbf{n}) = 0, \tag{8}$$

which have the componential forms

$$N' = H\theta' + p\eta, \tag{9}$$

$$H' = -N\theta' - pv, \tag{10}$$

$$M' = N\eta - Hv. \tag{11}$$

The material of the rod is taken to be elastic. Thus there are constitutive functions $\hat{N}$, $\hat{H}$, $\hat{M}$ such that

$$N(s) = \hat{N}(v(s), \eta(s), \mu(s), s), \text{ etc.} \tag{12}$$

We assume that $\hat{N}$, $\hat{H}$, $\hat{M}$ have as many derivatives as appear in the analysis. The common domain of these functions is restricted in an obvious way by (6). We require that:

$$\text{the matrix } \frac{\partial(\hat{N}, \hat{H}, \hat{M})}{\partial(v, \eta, \mu)} \text{ is positive-definite,} \tag{13}$$

$$\hat{N}(v, \eta, \mu, s) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \quad \text{as} \quad v \to \begin{Bmatrix} +\infty \\ V(\mu, s) \end{Bmatrix}, \tag{14}$$

$$\hat{H}(v, \eta, \mu, s) \to \pm\infty \quad \text{as} \quad \eta \to \pm\infty, \tag{15}$$

$$\hat{M}(v, \eta, \mu, s) \to \pm\infty \quad \text{as} \quad \mu \text{ approaches its right and left extremes of the region (6),} \tag{16}$$

$$\hat{N}(v, \cdot, \mu, s) \quad \text{and} \quad \hat{M}(v, \cdot, \mu, s) \text{ are even,} \quad \hat{H}(v, \cdot, \mu, s) \text{ is odd,} \tag{17}$$

$$\hat{N}(1, 0, \mathring{\theta}'(s), s) = 0, \quad \hat{M}(1, 0, \mathring{\theta}'(s), s) = 0. \tag{18}$$

Conditions (13)–(17) ensure that the constitutive mapping

$$(v, \eta, \mu) \mapsto (\hat{N}(v, \eta, \mu, s), \hat{H}(v, \eta, \mu, s), \hat{M}(v, \eta, \mu, s)) \tag{19a}$$

has the inverse

$$(N, H, M) \mapsto (\hat{v}(N, H, M, s), \hat{\eta}(N, H, M, s), \hat{\mu}(N, H, M, s)). \tag{19b}$$

It is easy to see that (19b) has symmetry conditions exactly analogous to (17).

If we insert the constitutive equations corresponding to (19b) into (9)–(11), (3) and (5), we obtain our governing differential equations:

$$N' = H\hat{\mu}(N,H,M,s)+p\hat{\eta}(N,H,M,s), \tag{20}$$

$$-H' = N\hat{\mu}(N,H,M,s)+p\hat{\nu}(N,H,M,s), \tag{21}$$

$$M' = N\hat{\eta}(N,H,M,s)-H\hat{\nu}(N,H,M,s), \tag{22}$$

$$\theta' = \hat{\mu}(N,H,M,s), \tag{23}$$

$$\mathbf{r}' = \hat{\nu}(N,H,M,s)\mathbf{a}(\theta)+\hat{\eta}(N,H,M,s)\mathbf{b}(\theta). \tag{24}$$

Had we substituted (12) directly into (9)–(11), then we would have obtained a quasilinear system of ordinary differential equations for the configuration and the strains. By using (19b), we obtain instead the semilinear system (20)–(24), which proves to be more convenient than the quasilinear system. (For dynamical problems, it is necessary to work with the quasilinear system.) One virtue of (19b) is that it readily accounts for the constraints of inextensibility and unshearability: An inextensible rod is characterized by the constitutive restriction that $\hat{\nu} = 1$, and an unshearable rod by $\hat{\eta} = 0$.

From (20)–(22) we obtain

$$H''+(N\hat{\mu}_{\mathrm{H}}+p\hat{\nu}_{\mathrm{H}})H'+(\hat{\mu}+N\hat{\mu}_{\mathrm{N}}+p\hat{\nu}_{\mathrm{N}})(H\hat{\mu}+p\hat{\eta})-(N\hat{\mu}_{\mathrm{M}}+p\hat{\nu}_{\mathrm{M}})(H\hat{\nu}-N\hat{\eta}) +N\hat{\mu}_{\mathrm{s}}+p\hat{\nu}_{\mathrm{s}} = 0, \tag{25}$$

$$\theta'' = \hat{\mu}_{\mathrm{N}}(H\hat{\mu}+p\hat{\eta})+\hat{\mu}_{\mathrm{H}}H'+\hat{\mu}_{\mathrm{M}}(N\hat{\eta}-H\hat{\nu})+\hat{\mu}_{\mathrm{s}}. \tag{26}$$

3. CONSTITUTIVE BEHAVIOR

We now regard our rod as a three-dimensional body with a natural reference configuration symmetrically disposed about the $\{\mathbf{i},\mathbf{j}\}$-plane. We restrict our attention to deformations that preserve this symmetry. We otherwise allow the reference configuration to be arbitrary, because it requires scarcely any more labor to treat this general case than it does to treat the case in which the reference configuration is a circular arch.

Notation

We employ Gibbsian notation: Second-order tensors (which are linear transformations of Euclidean 3-space into itself) are denoted by upper-case bold symbols. The value of tensor $\mathbf{A}$ at vector $\mathbf{v}$ is denoted $\mathbf{A}\cdot\mathbf{v}$ (in place of the more usual $\mathbf{Av}$) and the product of $\mathbf{A}$ and $\mathbf{B}$ is denoted $\mathbf{A}\cdot\mathbf{B}$ (in place of the more usual $\mathbf{AB}$). The transpose of $\mathbf{A}$ is denoted $\mathbf{A}^*$. We write $\mathbf{u}\cdot\mathbf{A} = \mathbf{A}^*\cdot\mathbf{u}$. The inner product of $\mathbf{A}$ and $\mathbf{B}$ (which equals the trace of $\mathbf{A}\cdot\mathbf{B}^*$) is denoted $\mathbf{A}:\mathbf{B}$. The dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted $\mathbf{ab}$ (in place of the more usual $\mathbf{a}\otimes\mathbf{b}$). It is defined by $(\mathbf{ab})\cdot\mathbf{u} = (\mathbf{b}\cdot\mathbf{u})\mathbf{a}$ for all $\mathbf{u}$. Thus $(\mathbf{ab}):(\mathbf{uv}) = (\mathbf{a}\cdot\mathbf{u})(\mathbf{b}\cdot\mathbf{v})$. Twice-repeated lower-case Latin indices are summed from 1 to 3 and twice-repeated lower-case Greek indices are summed from 1 to 2.

The first variation (Gâteaux differential) of $\mathbf{F}\mapsto\hat{\mathbf{T}}(\mathbf{F})$ at $\mathbf{G}$ in the direction $\mathbf{H}$ is $(\mathrm{d}/\mathrm{d}t)\hat{\mathbf{T}}(\mathbf{G}+t\mathbf{H})|_{t=0}$. When it is linear in $\mathbf{H}$ we denote this differential by $(\partial\hat{T}/\partial F)(\mathbf{G})\cdot\mathbf{H}$.

Let $(x^1,x^2,x^3 \equiv s)$ be curvilinear coordinates for the reference configuration of this body with the coordinate surface $x^2 = 0$ lying in the $\{\mathbf{i},\mathbf{j}\}$-plane. We regard s as the arc-length parameter of a base curve in this plane. Let $\mathring{\mathbf{p}}(x^1,x^2,s)$ be the reference position of the material point with coordinates (x^1,x^2,s). We introduce the basis $\{\mathbf{g}_k\}$, its dual $\{\mathbf{g}^k\}$, and the Jacobian j by

$$\mathbf{g}_k \equiv \frac{\partial\mathring{\mathbf{p}}}{\partial x^k}, \quad \mathbf{g}^k\cdot\mathbf{g}_l = \delta^k_l, \quad j \equiv (\mathbf{g}_1\times\mathbf{g}_2)\cdot\mathbf{g}_3, \tag{27}$$

where δ^k_l is the Kronecker delta. We henceforth denote partial derivatives with respect to

x^k by the subscript $,k$. We use the summation convention that twice repeated lower-case Latin indices are summed from 1 to 3, and twice repeated lower-case Greek indices are summed from 1 to 2. Fields obtained from $\mathring{\mathbf{p}}$ are likewise denoted by superposed circles.

Let $\mathbf{T}(x^1, x^2, s)$ and $\mathbf{S}(x^1, x^2, s)$ be the first and second Piola–Kirchhoff stress tensors at the material point with coordinates (x^1, x^2, s). They are related by $\mathbf{T} = \mathbf{F} \cdot \mathbf{S}$ where $\mathbf{F} = \mathbf{p}_{,k}\mathbf{g}^k$ is the deformation tensor. Let $\mathscr{A}(s_0)$ be the material surface defined by $s = s_0$ in the reference configuration of the body. Then the definition of $\mathbf{T}$ implies that the resultant contact force exerted across the section s is

$$\mathbf{n} = \int_{\mathscr{A}} \mathbf{T} \cdot \mathbf{g}^3 j \, dx^1 \, dx^2 = \int_{\mathscr{A}} \mathbf{T} \cdot \frac{\mathbf{g}^3}{|\mathbf{g}^3|} |\mathbf{g}_1 \times \mathbf{g}_2| \, dx^1 \, dx^2. \tag{28}$$

Here $\mathbf{T} \cdot \mathbf{g}^3$ denotes the value of the tensor (linear transformation) $\mathbf{T}$ at $\mathbf{g}^3$. Let $\mathbf{p}(x^1, x^2, s)$ be the deformed position of the material point with coordinates (x^1, x^2, s) and let $\mathbf{r}(s) \equiv \mathbf{p}(0, 0, s)$. Then the resultant contact torque about $\mathbf{r}(s)$ exerted across the section s is

$$M\mathbf{k} = \int_{\mathscr{A}} (\mathbf{p} - \mathbf{r}) \times \mathbf{T} \cdot \mathbf{g}^3 j \, dx^1 \, dx^2 = \int_{\mathscr{A}} (\mathbf{p} - \mathbf{r}) \times \mathbf{T} \cdot \frac{\mathbf{g}^3}{|\mathbf{g}^3|} |\mathbf{g}_1 \times \mathbf{g}_2| \, dx^1 \, dx^2. \tag{29}$$

One can show that the three-dimensional equilibrium equations imply that $\mathbf{n}$ and M defined by (28) and (29) satisfy (7) and (8) [cf. Antman (1976) and Antman and Marlow (1991)].

We want to use the representations (28) and (29) to motivate constitutive choices for (12). While we require that all constitutive functions be invariant under rigid motions, we do not always use forms of these functions that automatically guarantee this requirement. Let us accordingly constrain the position field $\mathbf{p}$ to have the form

$$\mathbf{p}(x^1, x^2, s) = \mathbf{r}(s) + x^1 \mathbf{b}(\theta(s)) + x^2 \mathbf{k}. \tag{30}$$

In this case the requirement that the deformation locally preserve orientation, which is expressed by the inequality that $(\mathbf{p}_{,1} \times \mathbf{p}_{,2}) \cdot \mathbf{p}_{,3} > 0$ everywhere (when the coordinates x^k are not singular), reduces to

$$v > \begin{Bmatrix} h_2(s)\mu & \text{if } \mu \geqslant 0 \\ h_1(s)\mu & \text{if } \mu \leqslant 0 \end{Bmatrix} \equiv V(\mu, s) \tag{31a}$$

where

$$\begin{Bmatrix} h_2(s) \\ h_1(s) \end{Bmatrix} \equiv \begin{Bmatrix} \max \\ \min \end{Bmatrix} \{x^1 : (x^1, x^2) \in \mathscr{A}(s)\},$$

which has the form (6). We require that $h_2 \mathring{\mu} < 1$ so that (31a) makes sense in the reference configuration. If $h_1 > 0$ or if $h_2 < 0$, then there are negative values of v compatible with (31a). This means that the base curve $\mathbf{r}$, which lies outside the body in these cases, could have portions with reversed orientation. We exclude this inconvenient possibility by requiring that

$$h_1 \leqslant 0 \leqslant h_2. \tag{31b}$$

For an elastic material the stress $\mathbf{T}$ can be decomposed as a sum of a *reactive stress* that does no virtual work in deformations of the form (30) and an *active* (or *extra*) stress that is prescribed by a constitutive function depending only on the gradient of (30). [The reactive stress may be thought of as "orthogonal" to the manifold defined by (30)]. It can be shown that the reactive stress makes no contribution to (28) and (29). [cf. Antman and Marlow (1991)]. Let $\hat{\mathbf{T}}$ and $\hat{\mathbf{S}}$ be the constitutive functions for active stresses. $\hat{\mathbf{T}}$ depends on $\mathbf{F}$ and (x^1, x^2, s) where

$$\mathbf{F} = [(v - x^1\mu)\mathbf{a} + \eta\mathbf{b}]\frac{\mathring{\mathbf{a}}}{1 - x^1\mathring{\mu}} + \mathbf{b}\mathring{\mathbf{b}} + \mathbf{k}\mathbf{k}, \tag{32}$$

and $\hat{\mathbf{S}}$ depends on (x^1, x^2, s) and

$$\mathbf{C} = \mathbf{F}^* \cdot \mathbf{F} = \frac{(v - x^1\mu)^2 + \eta^2}{(1 - x^1\mathring{\mu})^2}\mathring{\mathbf{a}}\mathring{\mathbf{a}} + \eta\frac{\mathring{\mathbf{a}}\mathring{\mathbf{b}} + \mathring{\mathbf{b}}\mathring{\mathbf{a}}}{1 - x^1\mathring{\mu}} + \mathring{\mathbf{b}}\mathring{\mathbf{b}} + \mathbf{k}\mathbf{k}. \tag{33}$$

We substitute these constitutive functions into (28) and (29) to obtain constitutive equations of the following form:

$$\begin{aligned}\hat{N} &= \mathbf{a} \cdot \int_{\mathscr{A}} \hat{\mathbf{T}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2 = \mathbf{a} \cdot \int_{\mathscr{A}} \mathbf{F} \cdot \hat{\mathbf{S}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2 \\ &= \int_{\mathscr{A}} \frac{v - x^1\mu}{1 - x^1\mathring{\mu}} \mathring{\mathbf{a}} \cdot \hat{\mathbf{S}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2, \end{aligned} \tag{34}$$

$$\begin{aligned}\hat{H} &= \mathbf{b} \cdot \int_{\mathscr{A}} \hat{\mathbf{T}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2 \\ &= \int_{\mathscr{A}} \left[\frac{\eta\mathring{\mathbf{a}}}{1 - x^1\mathring{\mu}} + \mathring{\mathbf{b}}\right] \cdot \hat{\mathbf{S}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2, \end{aligned} \tag{35}$$

$$\begin{aligned}\hat{M} &= \mathbf{k} \cdot \int_{\mathscr{A}} [x^1\mathbf{b} + x^2\mathbf{k}] \times \hat{\mathbf{T}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2 \\ &= -\int_{\mathscr{A}} x^1 \frac{v - x^1\mu}{1 - x^1\mathring{\mu}} \mathring{\mathbf{a}} \cdot \hat{\mathbf{S}} \cdot \mathring{\mathbf{a}}\, \mathrm{d}x^1\, \mathrm{d}x^2. \end{aligned} \tag{36}$$

These are special cases of (12). See the end of this section for examples.

The three-dimensional material is isotropic if and only if $\hat{\mathbf{S}}$ has the form

$$\hat{\mathbf{S}}(\mathbf{C}, x^1, x^2, s) = \alpha_0(\iota(\mathbf{C}), x^1, x^2, s)\mathbf{I} + \alpha_1(\iota(\mathbf{C}), x^1, x^2, s)\mathbf{C} + \alpha_2(\iota(\mathbf{C}), x^1, x^2, s)\mathbf{C} \cdot \mathbf{C}, \tag{37}$$

where $\iota(\mathbf{C})$ is the set of principal invariants of $\mathbf{C}$, which depend on $\det \mathbf{F} = (v - x^1\mu)/(1 - x^1\mathring{\mu})$ and $(\eta/(1 - x^1\mathring{\mu}))^2$, and where the αs are given scalar-valued functions. [The requirement that $\det \mathbf{F}$ be positive is the source of (31a).] We assume that $\hat{\mathbf{S}}(\mathbf{I}, x^1, x^2, s) = \mathbf{O}$ so that the reference configuration is stress free.

We could readily replace (30) with a more general expression. We refrain from doing so because the resulting complexity would merely interfere with our objective of exhibiting the effect of natural constitutive restrictions on the buckling process.

The construction of rod theories from a constraint such as (30) has been criticized from time to time in the literature on the grounds that the traction $\mathbf{T} \cdot \mathbf{n}$ on the lateral surface of the rod is completely determined by the constitutive function and is therefore unable to accommodate the traction prescribed in a given problem [cf. Novozhilov (1953)]. This objection is baseless, because the total traction is the sum of this active traction and that corresponding to the reactive stress, which is not prescribed constitutively. The presence of the reactive traction enables prescribed traction boundary conditions to be satisfied [cf. Podio-Guidugli (1989) and Antman and Marlow (1991)].

We now determine some general constitutive properties of elastic rods. From (32)–(36) we obtain

$$\alpha(\alpha\hat{N}_\nu+\beta\hat{N}_\eta+\gamma\hat{N}_\mu)+\beta(\alpha\hat{H}_\nu+\beta\hat{H}_\eta+\gamma\hat{H}_\mu)+\gamma(\alpha\hat{M}_\nu+\beta\hat{M}_\eta+\gamma\hat{M}_\mu)$$
$$=\int_{\mathscr{A}}[(\alpha-\gamma x^1)\mathbf{a}+\beta\mathbf{b}]\mathring{\mathbf{a}}:\frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{F}}:[(\alpha-\gamma x^1)\mathbf{a}+\beta\mathbf{b}]\mathring{\mathbf{a}}\frac{\mathrm{d}x^1\,\mathrm{d}x^2}{1-x^1\mathring{\mu}} \quad (38)$$

for arbitrary numbers α, β, γ. The Strong Ellipticity Condition implies that the integrand on the right-hand side of (38) is positive, so that the quadratic form on the left-hand side of (38) is positive-definite. This positive-definiteness is equivalent to (13) [to show this it is necessary to use a version of the Strong Ellipticity Condition adapted to constrained materials, see Antman and Marlow (1991)].

If we require the stresses to have suitable growth for **C** large and for **C** having some eigenvalues that are small, then we can deduce restrictions like (13)–(15). We do not pause to examine this question [see Antman (1976)].

We now study symmetry properties of the constitutive functions (34)–(36). These play a crucial role in our buckling analyses. $\mathscr{A}(s)$ is a region in the (x^1, x^2)-plane, which we are assuming to be symmetric about the line $x^2 = 0$. In consonance with all our symmetry assumptions on the deformation and shape, we require that the constitutive function $\hat{\mathbf{S}}$ be even in x^2. We term all these restrictions the *symmetry conditions*. Using (32)–(34), (36) and (37) we readily obtain:

Theorem 3.1. *Let the symmetry conditions hold and let* $\hat{\mathbf{S}}$ *be an isotropic function of* **C**. *Then*

$$\eta\mapsto\hat{H}(\nu,\eta,\mu,s)\ \textit{is odd},\quad \eta\mapsto\hat{N}(\nu,\eta,\mu,s),\quad \hat{M}(\nu,\eta,\mu,s)\ \textit{are even}. \quad (39)$$

This conclusion obviously holds even if the condition on isotropy is suitably relaxed, but we do not pause to demonstrate this. We can similarly obtain:

Theorem 3.2. *Let the symmetry conditions hold. If* $\mathscr{A}(s)$ *is symmetric about both the* x^1*- and* x^2*-axes and if* $\mathring{\mathbf{a}}\cdot\hat{\mathbf{S}}\cdot\mathring{\mathbf{a}}$ *is even in both* x^1 *and* x^2, *then*

$$\hat{M}(\nu,0,\mathring{\mu}\nu,s)=0. \quad (40)$$

If $\mathscr{A}(s)$ *is merely symmetric about the* x^1*-axis, with*

$$\int_{\mathscr{A}} x^1\,\mathrm{d}x^1\,\mathrm{d}x^2=0, \quad (41)$$

and if $\mathring{\mathbf{a}}\cdot\hat{\mathbf{S}}\cdot\mathring{\mathbf{a}}$ *is independent of* x^1 *and* x^2, *then* (40) *holds.*

The proof follows from (33) and (36) by straightforward computation.

Note that μ/ν is the curvature of **r** at any point at which $\eta = 0 = \eta'$. The positive-definiteness of (38) implies that $\hat{M}$ is a strictly increasing function of μ (but does not imply that $\hat{M}$ is a strictly increasing function of the actual curvature). In general, as Theorem 3.2 shows, $\hat{M}$ does not vanish when $\mu = \mathring{\mu}$. If $\mathring{\mu}$, which is the curvature of $\mathring{\mathbf{r}}$, is constant, so that $\mathring{\mathbf{r}}$ is a circle, then $\mu-\mathring{\mu}$ measures the departure from circularity. Thus we must expect that a purely inflational deformation taking $\mathring{\mathbf{r}}$ to a circle of different radius must be maintained by a nonzero bending couple $\hat{M}$. If we were to follow traditional practice and treat expressions like $x^1\mathring{\mu}$ and $x^1\mu/\nu$ as negligible with respect to 1 in (34) (i.e. if we were to treat the ratio of thickness to radius of curvature as small in both the reference and deformed configurations), then we would obtain a constitutive function $\hat{M}$ that vanishes when $\mu = \mathring{\mu}$. Such functions were used in most earlier works on arches [cf. Antman and Dunn (1980) and the references cited therein].

Note that the pressure is regarded as being applied to the material surface corresponding to $x^1 = 0$. A three-dimensional rod would have to have suitable cut-outs in order for the curve $x^1 = 0$ to pass through the centroid of $\mathscr{A}$. For the situation depicted in Fig. 1, it is reasonable to take the curve $x^1 = 0$ to correspond to $x^1 = \mathrm{h}_1$ by taking $h_1 = 0$. (Some of these issues can be avoided if the pressure is produced by a magnetic field acting on a current-carrying arch.) In view of these observations, we wish to generalize Theorem 3.2.

If we suspend the hypotheses of Theorem 3.2, we can nevertheless use assumptions (13) and (16) to show that the equation

$$\hat{M}(\nu, 0, \mu, s) = 0 \tag{42a}$$

has a unique solution μ in terms of the other variables, which we denote by

$$\mu = \mu^{\#}(\nu, s). \tag{42b}$$

It follows from (18) that $\mathring{\mu}(s) = \mu^{\#}(1, s)$. If $h_1 < 0$ and $h_2 > 0$, then the curve (42b) must lie between the lines $\nu = h_1\mu$ and $\nu = h_2\mu$. We can therefore assume that $\mu^{\#}(0, s) = 0$. (The behavior of the constitutive functions in a neighborhood of $(\nu, \mu) = (0, 0)$ is very singular.) If, however, $h_1 \geqslant 0$, the behavior of $\mu^{\#}$ is very different, as we shall show for the case that $h_1 = 0$, which meets the assumption (34b).

To investigate properties of $\mu^{\#}$, let us assume, primarily for simplicity of exposition, that $\mathring{\mu} = 1$ and that $\mathring{\mathbf{a}} \cdot \hat{\mathbf{S}} \cdot \mathring{\mathbf{a}}$ is independent of x^1 and x^2. These conditions are most natural for our buckling problems. From (36) and (39) we obtain an expression of the form

$$\hat{M}(\nu, 0, \mu, s) = -\int_{\mathscr{A}(s)} x^1 T\left(\frac{\nu - x^1\mu}{1 - x^1}, s\right) dx^1\, dx^2. \tag{43}$$

We assume that T is strictly increasing (which is a consequence of the Strong Ellipticity Condition) and that $T(1, s) = 0$ (which is a consequence of the requirement that the reference configuration be natural). Since

$$\hat{M}(\nu, 0, \nu, s) = -T(\nu, s) \int_{\mathscr{A}(s)} x^1\, dx^1\, dx^2, \tag{44}$$

it follows that $M(\nu, 0, \mu, s)$ has the same sign as $\int_{\mathscr{A}(s)} x^1\, dx^1\, dx^2$ on the line segment $\mu = \nu$, $\nu \in (0, 1)$. Consequently, if $\int_{\mathscr{A}(s)} x^1\, dx^1\, dx^2 > 0$, then the curve (42b) must be to the left of this segment in the (μ, ν)-plane (see Fig. 2). By setting (43) equal to zero, differentiating the resulting equation implicitly with respect to ν, we readily obtain

$$\mu_\nu^{\#}(\nu, s) = \frac{\displaystyle\int_{\mathscr{A}(s)} \frac{x^1}{1 - x^1} T'\left(\frac{\nu - x^1\mu^{\#}(\nu, s)}{1 - x^1}, s\right) dx^1\, dx^2}{\displaystyle\int_{\mathscr{A}(s)} \frac{(x^1)^2}{1 - x^1} T'\left(\frac{\nu - x^1\mu^{\#}(\nu, s)}{1 - x^1}, s\right) dx^1\, dx^2}. \tag{45}$$

Here the prime on T denotes its partial derivative with respect to its first argument. We

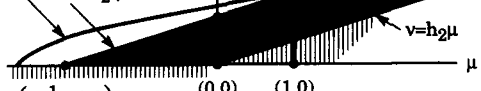

Fig. 2. Graph of (42) when $h_1 = 0$ and $\mathring{\mu} = 1$. The region of (μ, ν)-space corresponding to (31) lies above the lines $\nu = 0$ and $\nu = h_2\mu$. Condition (46) implies that $M(\nu, 0, \mu, s)$ assumes positive values in the shaded parallelogram. Note that under the hypotheses of Theorem 3.2, the graph of $\mu^{\#}$ lies on the ray from $(0, 0)$ to $(1, 1)$, and the region corresponding to (31) has a slightly different form. Earlier work by Antmann and Dunn (1980) posited that $\hat{M}$ vanishes on the line $\mu = 1$. (The corresponding figure for the excluded case that $h_1 > 0$ has a much different character.)

note that $\mu_v^\#(v, s) = 0$ only under very special circumstances (see the comments in the caption of Fig. 2).

Let us now treat the case that $h_1 = 0$. From (45) we immediately obtain

$$\mu_v^\#(v, s) > 1/h_2. \tag{46}$$

Thus $M(v, 0, \mu, s)$ is positive on the parallelogram bounded by the lines $v = 0$, $v = 1$, $v = h_2\mu$, $v = 1+h_2(\mu-1)$. Hence the curve (42b) must be to the left of the line $v = 1+h_2(\mu-1)$ for $0 \leqslant v \leqslant 1$ and must lie between the lines $v = 1$ and $v = 1+h_2(\mu-1)$ for $\mu \geqslant 1$ (see Fig. 2). We easily obtain an upper bound for $\mu_v^\#(v, s)$ depending on T:

$$\mu_v^\#(v, s) \leqslant \frac{\max_{0 \leqslant x^1 \leqslant h_2}\left\{T'\left(\dfrac{v - x^1\mu^\#(v, s)}{1-x^1}, s\right)\right\} \displaystyle\int_{\mathscr{A}(s)} \frac{x^1}{1-x^1}\, dx^1\, dx^2}{\min_{0 \leqslant x^1 \leqslant h_2}\left\{T'\left(\dfrac{v - x^1\mu^\#(v, s)}{1-x^1}, s\right)\right\} \displaystyle\int_{\mathscr{A}(s)} \frac{(x^1)^2}{1-x^1}\, dx^1\, dx^2}. \tag{47}$$

We shall limit our attention exclusively to the cases in which the hypotheses of Theorem 3.2 hold or in which $h_1 = 0$, both because these cases are of the most physical interest and because they typify the richness of other choices.

Examples

It is easy to construct specific constitutive functions for rods from those for three-dimensional bodies: Suppose that the three-dimensional body is isotropic and homogeneous, and therefore has a strain-energy function Ω depending only on $\iota(\mathbf{C})$. Then the rod theory generated by (30) has the strain-energy function

$$W(v, \eta, \mu) = \int_{\mathscr{A}} \Omega(\iota(\mathbf{C}))(1 - x^1\mathring{\mu})\, dx^1\, dx^2. \tag{48}$$

with $\hat{N} = W_v$, $\hat{H} = W_\eta$, $\hat{M} = W_\mu$ [cf. Antman (1976)]. Now suppose further that $\mathscr{A}$ is the rectangle $h_1 < x^1 < h_2$, $-(t/2) < x^2 < (t/2)$ and that $\Omega(\iota(\mathbf{C}))$ has the form

$$\Omega(\iota(\mathbf{C})) = A\left(\frac{v - x^1\mu}{1 - x^1\mathring{\mu}}\right)^{-a} + B\left(\frac{v - x^1\mu}{1 - x^1\mathring{\mu}}\right)^{b} + C\left|\frac{\eta}{1 - x^1\mathring{\mu}}\right|^{c}$$
$$+ D\left(\frac{v - x^1\mu}{1 - x^1\mathring{\mu}}\right)^{-d}\left|\frac{\eta}{1 - x^1\mathring{\mu}}\right|^{f} + E\left(\frac{v - x^1\mu}{1 - x^1\mathring{\mu}}\right)^{e}\left|\frac{\eta}{1 - x^1\mathring{\mu}}\right|^{g}, \tag{49}$$

where $A, \ldots, E$, $a, \ldots, g$ are given constants with A, B, C, $a+1$, $b-1$, $c-1 > 0$ and with the remaining constants adjusted so that (14)–(16) hold. We can include in (49) further expressions of the same form. If a, b, d, e, f are integers, then we can compute W of (48) from (49) in closed form. In particular, for $a = 1$, $b = 2$, $c \neq 2$, $D = 0 = E$, and $\mathring{\mu} = 1$, we obtain

$$W = \frac{At}{\mu^3}\left[(v-\mu)^2 \ln\left(\frac{v - h_1\mu}{v - h_2\mu}\right) + (h_2 - h_1)(2\mu^2 - v\mu) - \tfrac{1}{2}(h_2 - h_1)^2\mu^2\right]$$
$$+ Bt\left[(v-\mu)^2 \ln\left(\frac{1-h_1}{1-h_2}\right) + 2(h_2 - h_1)(v-\mu)\mu + (h_2 - h_1)(1 - \tfrac{1}{2}h_1 - \tfrac{1}{2}h_2)\mu^2\right]$$
$$+ \frac{Ct}{c-2}[(1-h_2)^{2-c} - (1-h_1)^{2-c}]|\eta|^c. \tag{50}$$

Three applications of l'Hôpital's rule show that the coefficient of $\mathbf{A}$ is regular at $\mu = 0$. For

$c \neq 2$, the coefficient of C does not behave properly at $\eta = 0$. For $c > 2$, this difficulty is easily handled by adding to (50) a term quadratic in η. For $c < 2$, a somewhat more sophisticated modification is required. In this case, we should regard (50) as characterizing the behavior of energy only for large $|\eta|$.

By differentiating (50) with respect to μ, we obtain the quite complicated expression corresponding to (43) and the much simpler expression corresponding to (44). On the other hand, if we differentiate the integral of (49) with respect to μ, we recover (43) directly, from which we can obtain T. Then we can compute (45) in closed form. The complexity of the resulting expressions illuminates the virtues of our approach of using general constitutive equations.

4. BOUNDARY-VALUE PROBLEMS FOR THE ARCH

We now specialize the preceding considerations to circular arches scaled to have unit radius: $\mathring{\mu} = 1$. We take $-\alpha \leqslant s \leqslant \alpha$ with $0 < \alpha < \pi$. We assume that the material of the arch is homogeneous and that its cross-section is constant, so that $\mathscr{A}$ is independent of s. Thus the constitutive functions are independent of s. The reference configuration is defined by

$$\mathring{\mathbf{r}}(s) = \sin s\mathbf{i} - \cos s\mathbf{j}, \quad \mathring{\theta}(s) = s. \tag{51}$$

In order to ensure that our extensible arches admit a one-parameter family of circular (trivial) equilibrium states (see Section 5) we assume that the ends $s = \pm\alpha$ of the arch are attached to blocks that can slide without friction in grooves that are symmetrically disposed about the $\mathbf{j}$-axis. We assume that these grooves have graphs of the form

$$x = \pm f(y) \quad \text{or equivalently} \quad y = g(\pm x), \tag{52}$$

where f and $g = f^{-1}$ are smooth functions. We assume that f and g are respectively defined on corresponding intervals containing $\cos\alpha$ and $\sin\alpha$, which are the natural reference values of $y(\alpha)$ and $x(\alpha)$ (see Fig. 1). We assume that f' and g' are everywhere negative on their domains. For our global analysis it is more convenient to use f and for our local analysis it is more convenient to use g.

The corresponding position boundary conditions are

$$x(\pm\alpha) = \pm f(y(\pm\alpha)) \quad \text{or} \quad y(\pm\alpha) = g(\pm x(\alpha)). \tag{53}$$

If the ends are subject to no further restrictions, then the absence of friction in the grooves implies that there are no components of contact forces in the directions tangent to the grooves:

$$\mathbf{n}(\pm\alpha)\cdot[\pm f'(y(\pm\alpha))\mathbf{i}+\mathbf{j}] = 0 \quad \text{or} \quad \mathbf{n}(\pm\alpha)\cdot[\mathbf{i}\pm g'(\pm x(\pm\alpha))\mathbf{j}] = 0. \tag{54}$$

If the end $s = \pm\alpha$ is hinged, then

$$M(\pm\alpha) = 0. \tag{55}$$

If the end $s = \pm\alpha$ is welded to the block, then $\mathbf{b}(\theta(\pm\alpha))$ must be tangent to the groove. Thus $\mathbf{a}(\theta(\pm\alpha))$ must be perpendicular to the groove:

$$\mathbf{a}(\theta(\pm\alpha))\cdot[\pm f'(y(\pm\alpha))\mathbf{i}+\mathbf{j}] = 0 \quad \text{or} \quad \mathbf{a}(\theta(\pm\alpha))\cdot[\mathbf{i}\pm g'(\pm x(\pm\alpha))\mathbf{j}] = 0. \tag{56a}$$

Combining this with (54) we deduce that

$$H(\pm\alpha) = 0. \tag{56b}$$

Because of the freedom afforded by the presence of the grooves, when the arch is hinged at both ends, there can be a multitude of skewed trivial solutions and corresponding nontrivial solutions for a given p. We can avoid some of the consequent difficulties for this and other bifurcation problems by imposing the constraint that both ends of the arch be held at the same height:

$$y(-\alpha) = y(\alpha) \equiv \bar{y}. \tag{57}$$

It then follows from (53) that $-x(-\alpha) = x(\alpha) \equiv \bar{x}$. The imposition of (57) brings our problems, in which the ends slide in grooves, closer to the classical problems for inextensible arches, in which the ends are fixed. For extensible arches, we cannot avoid using grooves if we wish to have problems with natural trivial (unbuckled) states.

Such a set of boundary conditions can be enforced by confining the ends of the arch to lie not only in the grooves described above, but also in the horizontal slot illustrated in Fig. 3. A constraint such as (57) is typically maintained by reactions. We study these and the boundary conditions they induce below.

We determine the appropriate boundary conditions to replace (54) when (57) holds by carefully analysing the free-body diagrams for the arch and the framework of Fig. 3: Let the reactions of the grooves of Fig. 1 at $\pm\alpha$, which are perpendicular to the groove, be denoted $A^{\pm}[\mp g'(\bar{x})\mathbf{i}+\mathbf{j}]$. Since the only vertical forces acting on the apparatus of Fig. 3 are those exterted by the ends of the arch, and since the apparatus exerts no horizontal forces on the arch, we can take the reactions at $\pm\alpha$ of the apparatus on the arch to have the form $\pm B\mathbf{j}$. The total force on the arch is therefore

$$\int_{-\alpha}^{\alpha} p\mathbf{k}\times\mathbf{r}'\,\mathrm{d}s + A^{+}[-g'(\bar{x})\mathbf{i}+\mathbf{j}] + A^{-}[g'(\bar{x})\mathbf{i}+\mathbf{j}]. \tag{58}$$

Setting it equal to $\mathbf{0}$ and using (57), we obtain

$$A^{+} = A^{-} \equiv A = -p\bar{x}. \tag{59}$$

The total torque on the arch about $\mathbf{0}$ is therefore

$$\mathbf{r}(\alpha)\times[-p\bar{x}(-g'\mathbf{i}+\mathbf{j})+B\mathbf{j}] + \mathbf{r}(-\alpha)\times[-p\bar{x}(g'\mathbf{i}+\mathbf{j})-B\mathbf{j}] + p\int_{-\alpha}^{\alpha}\mathbf{r}\times(\mathbf{k}\times\mathbf{r}')\,\mathrm{d}s + [M(\alpha)-M(-\alpha)]\mathbf{k}. \tag{60}$$

Setting it equal to $\mathbf{0}$ and using (57), we obtain

$$2B\bar{x} + M(\alpha) - M(-\alpha) = 0. \tag{61}$$

If the arch is hinged at both ends, then $M(\pm\alpha) = 0$ and (61) implies that $B = 0$. Thus

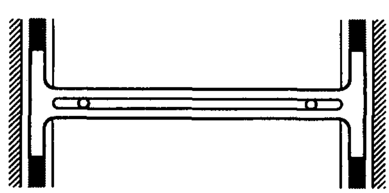

Fig. 3. Apparatus keeping the ends of an arch at the same height. The ends of the arch are confined both to the frictionless grooves shown in Fig. 1 and to the frictionless horizontal sleeve shown here. The vertical segments of the apparatus slide in frictionless vertical grooves.

for the doubly hinged arch, the apparatus of Fig. 3 exerts no forces on the ends of the arch. Thus conditions (54) remain valid. More generally, from (59), (62), and the use of the sign convention, we obtain

$$\mathbf{n}(\pm\alpha) = p\bar{x}[g'(\bar{x})\mathbf{i}\mp\mathbf{j}]+\frac{1}{2\bar{x}}[M(-\alpha)-M(\alpha)]\mathbf{j} \tag{62}$$

(which is equivalent to (54) when $M(\pm\alpha) = 0$).

The boundary-value problem

The governing equations for our boundary-value problems are (3), (20)–(22). The boundary conditions for arches not supported by the apparatus of Fig. 3 consist of (53), (54), and one condition at each end from (55) and (56). For arches supported by the apparatus, the boundary conditions are (53), (57), one condition at each end from (55) and (56a), and (62).

We observe that there are exactly six scalar boundary conditions for any arch not supported by the apparatus. This number of boundary conditions corresponds to the order of the governing equations. On the other hand, for an arch supported by the apparatus, we have three boundary conditions from (53), (57), two conditions from the allowable choices from (55) and (56a), together with four conditions from (62). This fact causes no difficulty for our global analysis because the redundant boundary conditions implicit in (62) merely correspond to integrals of the governing equations. It does, however, cause trouble in our local analysis, because we have to compute the adjoint of an ordinary differential operator. Thus we wish to select a suitable single condition from (62). To motivate our choice we compute the virtual work of the forces at the ends of the arch. The virtual displacements corresponding to (53), (57) have the form $\delta\mathbf{r}(\pm\alpha) = [\mathbf{i}\pm g'(\bar{x})\mathbf{j}]\delta x(\alpha)$ where $\delta x(\alpha)$ is arbitrary. Thus the corresponding virtual work is

$$\mathbf{n}(\alpha)\cdot\delta\mathbf{r}(\alpha)-\mathbf{n}(-\alpha)\cdot\delta\mathbf{r}(-\alpha) = \{\mathbf{n}(\alpha)\cdot[\mathbf{i}+g'(\bar{x})\mathbf{j}]+\mathbf{n}(-\alpha)\cdot[\mathbf{i}-g'(\bar{x})\mathbf{j}]\}\delta x(\alpha). \tag{63}$$

The coefficient of $\delta x(\alpha)$ in (63) is the generalized force corresponding to $\bar{x}$. We substitute (62) into this expression for the generalized force and find that it vanishes:

$$\mathbf{n}(\alpha)\cdot[\mathbf{i}+g'(\bar{x})\mathbf{j}]+\mathbf{n}(-\alpha)\cdot[\mathbf{i}-g'(\bar{x})\mathbf{j}] = 0. \tag{64}$$

We adopt this scalar equation as a replacement for the system (62); accordingly, in our statement of boundary conditions for arches supported by the apparatus, we can replace (62) with (64).

5. TRIVIAL STATES

We say that an equilibrium configuration is *trivial* if there is no shear and if $\mathbf{r}$ describes a circle. Thus in such a state $\eta = 0$, or equivalently, $H = 0$ by virtue of (17), and the constant curvature $\theta'(v^2+\eta^2)^{-1/2}+(v\eta'-\eta v')(v^2+\eta^2)^{-3/2}$ reduces to θ'/v. We limit our attention to positive values of p. One purpose of this section is to determine the groove shapes f (or g) for which there are trivial solutions.

Setting $H = 0$ in (20)–(22) we obtain

$$N = \text{const.}, \quad M = \text{const.}, \tag{65a,b}$$

$$N\hat{\mu}(N, 0, M) = -p\hat{v}(N, 0, M). \tag{66}$$

From (65) it follows that v and μ are constant, so that the fact that $\mathbf{r}$ is circular in the trivial state is a consequence of the assumption that $H = 0$. We thus have that

$$\theta = \mu s + c, \tag{67}$$

where c is a constant to be determined.

We restrict our attention to a continuous family of trivial states parametrized by $p \geqslant 0$. Since the right-hand side of (66) is negative, so is the left. When p is small and positive, μ is positive by (18) and (19), so that N is negative. Since the left-hand side of (66) can never vanish, we conclude that N is everywhere negative and μ is everywhere positive in such a continuous 1-parameter family of trivial states.

Since $N < 0$ when $p > 0$, it follows that for trivial solutions, (54) reduces to (56):

$$f'(y(\pm\alpha)) \cos \theta(\pm\alpha) \pm \sin \theta(\pm\alpha) = 0 \quad \text{or} \quad \cos \theta(\pm\alpha) \pm g'(\pm x(\pm\alpha)) \sin \theta(\pm\alpha) = 0. \tag{68}$$

Equation (67) reduces (68) to

$$\tan c = \pm \frac{f'(y(\pm\alpha)) \cos \mu\alpha + \sin \mu\alpha}{f'(y\pm\alpha)) \sin \mu\alpha - \cos \mu\alpha}. \tag{69}$$

From (3) and (53) we get

$$\begin{aligned} \mathbf{r}(\alpha) - \mathbf{r}(-\alpha) &= [f(y(\alpha)) - f(y(-\alpha))]\mathbf{i} + [y(\alpha) - y(-\alpha)]\mathbf{j} \\ &= v \int_{-\alpha}^{\alpha} [\cos(\mu s + c)\mathbf{i} + \sin(\mu s + c)\mathbf{j}]\, \mathrm{d}s \\ &= 2\frac{v}{\mu} \sin \mu\alpha [\cos c\mathbf{i} + \sin c\mathbf{j}]. \end{aligned} \tag{70}$$

Doubly-welded arch

For a doubly-welded arch with given g, the system (66), (69), (70), subject to the constitutive equations, is a set of five scalar equations for the five unknowns N, M, c, $y(\alpha)$, $y(-\alpha)$. Each such solution of this "determinate" system generates a trivial state of our problem. For each given f, we can seek a family of trivial states parametrized by p. Avoiding sterile generality, we content ourselves with treating this problem for radial grooves, for which $f'(y) = -\tan \alpha$. Thus (69) yields $\tan c = \mp \tan(\mu - 1)\alpha$, so that $c = 0$ and $\mu = 1$. Combining this last condition with (66), we obtain

$$N + p\hat{v}(N, 0, M) = 0, \quad \hat{\mu}(N, 0, M) = 1, \tag{71a,b}$$

which has a unique solution for N and M when $p \geqslant 0$, as a consequence of (13). Note that (71b) says that there is no change in the bending strain μ. (If (40) holds, then we can identify this solution as $N = -p$, $M = 0$.) In fact, if we merely assume that the grooves are rays, and not impose (57), then (69) forces $c = 0$, which implies the symmetry condition. Below we discuss the case in which symmetry is imposed, but the grooves are not radial.

Arch with at least one end hinged

On the other hand, if at least one condition from (55) should hold, then (65) implies that $M = 0$. In this case, system (66), (69), (70) yields a system of five scalar equations for the four unknowns $N, c, y(\alpha)$, $y(-\alpha)$, which is unlikely to have a solution. To circumvent this difficulty, we shall seek groove shapes f for which this system can be solved for each $p > 0$. Observe that radial grooves cannot be expected to suffice: In this case (71) holds with the additional restriction that $M = 0$. Indeed, (71a) implies that $N < 0$ for $p > 0$, so that $\hat{v}(N, 0, 0) < 1$. But if $\int_{\mathscr{A}} x^1\, \mathrm{d}x^1\, \mathrm{d}x^2 \geqslant 0$, then the analysis leading to Fig. 2 shows that μ must be less than 1, a contradiction. More generally, whenever $\mu_v^{\#}(1) \neq 0$, our overdetermined system cannot hold for radial grooves in a neighborhood of the reference configuration.

Symmetric trivial states

From now on we define a *symmetric* configuration to be one that is symmetric about the y-axis. For trivial states satisfying (57), we obtain from (62) that $B = 0$ since M is constant. Thus the forces maintaining (57) are zero, and (63) reduces to (54). Consequently, it makes no difference whether we seek trivial solutions of the unconstrained problem that happen to satisfy (57) or we seek solutions satisfying (53), (57), one condition at each end from (55) and (56), and (63). It follows that every trivial solution must satisfy (54).

Let us first suppose that f is prescribed. If trivial solutions are constrained by (57), then $c = 0$, and the governing system (66), (69), (70) reduces to (66) and

$$\tan \mu\alpha = -f'(\bar{y}), \quad \bar{x} = f(\bar{y}) = \frac{v}{\mu} \sin \mu\alpha. \tag{72a,b}$$

Equations (66), (72), and the constitutive equations provide five equations for the five unknowns N, M, v, μ, $\bar{y}$. For a doubly-welded arch, there are no further restrictions. Let us pause to study the solvability of this system. We assume that this reduced system is satisfied in the reference configuration, i.e. that f is such that $f'(\cos \alpha) = -\tan \alpha$. We look for a connected family of pairs of solutions and parameters p of (66), (72) that contains the reference configuration for $p = 0$. We accordingly use the principal branch of arctan to solve (72) for μ in terms of $\bar{y}$:

$$\mu\alpha = -\arctan f'(\bar{y}) \tag{73}$$

and then use (66) and (70b) to express N in terms of $\bar{x}$:

$$N = -\frac{pf(\bar{y})}{\sin \mu\alpha} = \frac{pf(\bar{y})\sqrt{1+f'(\bar{y})^2}}{f'(\bar{y})}. \tag{74}$$

We substitute (73) and (74) into (66) to obtain

$$\frac{N}{p}\alpha\mu = -\frac{f(\bar{y})\sqrt{1+f'(\bar{y})^2}}{f'(\bar{y})} \arctan (f'(\bar{y})) = \alpha\hat{v}(N, 0, M) = \alpha\hat{v}\left(\frac{pf(\bar{y})\sqrt{1+f'(\bar{y})^2}}{f'(\bar{y})}, 0, M\right), \tag{75}$$

where M is the solution of

$$\alpha\hat{\mu}\left(\frac{pf(\bar{y})\sqrt{1+f'(\bar{y})^2}}{f'(\bar{y})}, 0, M\right) = -\arctan f'(\bar{y}). \tag{76}$$

This solution for M exists and is unique by virtue of (13), (16). The substitution of this solution into (75) converts it to an equation for $\bar{y}$ when p and f' are prescribed. Thus for each p there are as many trivial solutions as there are solutions $\bar{y}$ of its equation. Rather than studying this difficult equation, we again restrict the problem for the doubly-welded arch to that for radial grooves.

Let us now treat the case in which at least one end, say $s = -\alpha$, is hinged. Thus (55) holds, so that $M = 0$. This condition, appended to the governing equations, gives six equations for the same five unknowns. We accordingly do not prescribe f, but rather seek those fs for which this system is solvable.

Since $M = 0$, it follows from (42) that $\mu = \mu^{\#}(v)$. The substitution of this formula into (70a) yields

$$-f'(\bar{y}) = \tan(\alpha\mu^{\#}(v)). \tag{77}$$

In place of (75) we likewise get

$$-\frac{f(\bar{y})\sqrt{1+f'(\bar{y})^2}}{f'(\bar{y})}\arctan(f'(\bar{y})) = \alpha v, \tag{78}$$

the substitution of which into (77) yields an implicit equation for f':

$$-f'(\bar{y}) = \tan\left(\alpha\mu^{\#}\left(-\frac{f(\bar{y})\sqrt{1+f'(\bar{y})^2}}{\alpha f'(\bar{y})}\arctan(f'(\bar{y}))\right)\right), \tag{79}$$

which is independent of p. The unique solvability of this equation for $f'(\bar{y})$ is by no means obvious. For those ranges of $\bar{y}$ for which it lacks a solution there can be no pair of grooves giving a trivial solution. We now investigate this question.

If the hypotheses of Theorem 3.2 hold, then $\mu^{\#}(v) = v$. Therefore $N = -p$ and $\mu = v = \hat{v}(-p, 0, 0)$, so that (79) reduces to

$$f' = -\frac{f}{\sqrt{1-f^2}} \quad \text{or} \quad g'(\bar{x}) = 2\frac{\sqrt{1-\bar{x}^2}}{\bar{x}}, \tag{80}$$

which implies that

$$g(\bar{x}) = -\ln\bar{x} + \ln(1+\sqrt{1+\bar{x}^2}) - \sqrt{1-\bar{x}^2} + \text{const}, \quad \text{for } 0 < \bar{x} < 1. \tag{81}$$

The dominant term in (81) is $-\ln\bar{x}$.

Let us now study (79) when $h_1 = 0$, in which case the hypotheses leading to (80) fail. The solvability of (79) when $\bar{y}$ and $f'(\bar{y})$ are near their reference values $\cos\alpha$ and $f'(\cos\alpha) = -\tan\alpha$ is assured by the Implicit Function Theorem provided that

$$1 + \mu_v^{\#}(1)\sin\alpha(\cos\alpha - \alpha^{-1}\sin\alpha) \neq 0. \tag{82}$$

Now (45) says that $\mu_v^{\#}(1)$ exceeds 1, possibly by a large amount. [We readily get an explicit representation for $\mu_v^{\#}(1)$ by specializing (44).] Thus the left-hand side of (82) is positive if α is taken sufficiently small. On the other hand, we can make the left-hand side of (82) negative by taking α sufficiently large and by taking h_2 sufficiently small.

Now let us study the global solvability of (79) when $h_1 = 0$. Let $\alpha\omega \equiv -\arctan f'(\bar{y}) \equiv \operatorname{arccot}(-g'(\bar{x}))$ and let $v^{\#}$ be the inverse of $\mu^{\#}$. The existence of this inverse is ensured by (45). Then (79) reduces to

$$\frac{f(\bar{y})}{\alpha} = \frac{\bar{x}}{\alpha} = \frac{\sin\alpha\omega}{\alpha\omega}v^{\#}(\omega) \equiv F(\omega). \tag{83}$$

Solving this equation for ω is equivalent to solving (79) for f' in terms of f or solving (79) for g' in terms of $\bar{x}$. Since we want solutions with f', we restrict ω to the interval $[0, (\pi/2\alpha)]$. Since

$$F(0) = v^{\#}(0), \quad F\left(\frac{\pi}{2\alpha}\right) = \frac{2}{\pi}v^{\#}\left(\frac{\pi}{2\alpha}\right),$$

we use Fig. 2 to find that

$$1-h_2 < F(0) < 1, \quad \frac{2}{\pi} < F\left(\frac{\pi}{2\alpha}\right) < \frac{2}{\pi}\left[1+h_2\left(\frac{\pi}{2\alpha}-1\right)\right].$$

By taking h_2 to be sufficiently small, we can ensure that $F(0) > F(1)$. Then F is invertible if it satisfies the constitutive restriction that $F'(\omega) < 0$ for $0 \leqslant \omega \leqslant (\pi/2\alpha)$, provided that $\bar{x}/\alpha$ is restricted to the range $[F(\pi/2\alpha), F(0)]$. Thus $\bar{x} = f(\bar{y})$ is necessarily bounded below by $2\alpha/\pi$. (It might also be possible to have solutions when F' is everywhere positive for certain parameter ranges.) Once ω is found, we can in principle immediately construct g' and then g by integration or construct f by quadratures. Moreover, this formulation shows that where it exists f' must be negative, as desired. This analysis shows how delicately the solvability of (79) depends on the thickness $h_2 - h_1$, the angle α, the constitutive functions, and the location of the material curve $x^1 = 0$, on which the pressure is applied, relative to the curve of centroids of the arch.

Note that the f' of (80) is independent of α, whereas the f' of (79) typically depends on α. In particular, if (40) were replaced with $\mu = 1$, then f would describe a radial groove making angle α with the vertical. That f is typically not radial is a direct consequence of the extensibility of the arch. (For inextensible circular arches, the trivial solution just corresponds to the reference state, so that the ends can be fixed.)

Wherever f can be found from (79), we assume that the constitutive restrictions are such that the equation

$$N\hat{\mu}(N,0,0) = -p\hat{v}(N,0,0) \tag{84}$$

has a unique solution for N in terms of p whenever p is positive (or else we restrict our attention to ranges of p or N for which (84) has a solution). To see what is involved, it is only necessary to sketch the left- and right-hand sides of (84) as functions of N, noting that $\hat{\mu}(0,0,0) = 1$. In particular, if $N \mapsto \hat{\mu}(N,0,0)$ has a positive lower bound for $N \leqslant 0$, then (84) has at least one solution for each $p > 0$, and if furthermore $N \mapsto \hat{\mu}_N(N,0,0)$ has a positive lower bound for $N \leqslant 0$, then (84), has exactly one solution for each $p > 0$.

When (80) holds, we immediately compute that $f'' > 0$, i.e. that f and therefore g are convex. To check where f is convex for (79) we compute from it that

$$\left[1 - \frac{f(f' - \arctan f')}{(f')^2}\mu_v^{\#}\right] f'' = -[1+(f')^2] \arctan (f')\mu_v^{\#}, \tag{85}$$

where the argument of $\mu_v^{\#}$ is the same as that of $\mu^{\#}$ in (79). Since the right-hand side of (85) is positive and the bracketed expression on the left is positive, it follows that f is convex wherever it is defined.

It is illuminating to compare our treatment of these trivial states with those for a doubly-hinged arch with fixed ends, for which the boundary conditions are

$$M(\pm\alpha) = 0, \quad x(\pm\alpha) = \pm\sin\alpha, \quad y(\pm\alpha) = -\cos\alpha. \tag{86}$$

In this case we find that N must satisfy the overdetermined system comprising (84) and

$$\hat{\mu}(N,0,0) \sin\alpha = \hat{v}(N,0,0) \sin(\hat{\mu}(N,0,0)\alpha). \tag{87}$$

In other words, we get a family of trivial states only for those materials that satisfy the constitutive restriction (87) identically. This condition seems natural only for inextensible arches, which are defined by $\hat{v}(N,0,0) \equiv 1$, for which $\hat{\mu}(N,0,0)$ is independent of N (so that $\hat{\mu}(N,0,0) = 1$).

6. GLOBAL PROPERTIES OF BUCKLED STATES

Doubly-welded arch

We now study the global behavior of buckled states of a doubly-welded arch whose ends move freely in radial grooves. Throughout this section we assume that $p > 0$. The equilibrium is governed by the differential equations (20)–(25) and the boundary conditions specializing (54), (56), and (53):

$$H(\pm\alpha) = 0, \tag{88}$$

$$\theta(\pm a) = \pm\alpha, \tag{89}$$

$$\mathbf{r}(\pm\alpha)\cdot(\cos\alpha\mathbf{i} \pm \sin\alpha\mathbf{j}) = 0. \tag{90}$$

It follows from the analog of (17) for (19b) that (25) admits the solution $H = 0$ no matter what functions N and M and what parameters p appear in (25). Let us define a (*solution*) *branch* of our boundary-value problem to be any connected set of $((N, H, M, \theta, \mathbf{r}), p)$ satisfying integral equations corresponding to the boundary-value problem with $(N, H, M, \theta, \mathbf{r})$ continuous. In view of our blanket smoothness assumption on the constitutive functions, it follows from (20)–(24) that such a quintuple $(N, H, M, \theta, \mathbf{r})$ is twice continuously differentiable, so that (25) is satisfied in a classical sense. The existence of a global branch emanating from an eigenvalue of odd algebraic multiplicity of the linearization of the boundary-value problem about the trivial solution is assured by the theory of Rabinowitz (1971).

The most important global feature of solutions of this boundary-value problem is that the number of simple zeros of H is constant on any branch not containing a trivial solution. (A *simple* zero is one at which H' does not vanish.) The justification of this assertion [cf. Crandall and Rabinowitz (1970)] is based on a precise formulation of the observation that in view of (88), the only way H could change the number of its simple zeros would be for it to have double zero. Thus there would be an s_0 such that $H(s_0) = 0 = H'(s_0)$. For given N, M, p, the initial-value problem consisting of (25) subject to these initial conditions has a unique solution, which we immediately identify as the trivial solution $H = 0$. This solution generates the trivial state with (71) holding. In view of this argument, we can globally distinguish different nontrivial branches of solutions of this boundary-value problem by the nodal properties of the function H.

For doubly-welded arches with horizontal ends we consider only problems with radial grooves. From (62) we obtain

$$H(\pm\alpha) = \frac{\cos\alpha}{2\bar{x}}[M(-\alpha) - M(\alpha)] = \frac{\cot\alpha}{2}[N(\alpha) - N(-\alpha)]. \tag{91}$$

Let us restrict our attention to symmetrical equilibrium states, for which x is an odd function, y is an even function, and θ is an odd function. Then (17) implies that H is odd. Under these conditions (91) implies that $H(\pm\alpha) = 0$. Therefore the only way for H to change the number of its zeros is for it to have a double zero, in which case the solution is trivial. Thus nontrivial symmetric solution branches are globally distinguished by the nodal properties of H.

The treatment of other boundary conditions is far more delicate. We just sketch the main ideas for the doubly-hinged arch, with or without its ends constrained to be horizontal, and indicate the source of difficulty. Condition (55) holds. Now H can change its nodal properties by having a double zero at a point of the open interval $(-\alpha, \alpha)$, or by having a double zero at an end, where M is prescribed equal to zero. In the first case, we know that H is identically 0. Therefore let us study the case in which

$$H(-\alpha) = 0 = M(-\alpha). \tag{92}$$

It follows from (17), (20), (22) that $N'(-\alpha) = 0 = M'(-\alpha)$. A solution of the boundary-value problem consisting of (the autonomous versions of) (20)–(22) subject to (92) and

(55) corresponds to a trajectory in the (N, H, M) phase space starting on the N-axis. Condition (17) says that the phase portrait is symmetric about the plane $H = 0$. The symmetry of the phase portrait implies that

$$H(\pm\alpha) = 0 = M(\pm\alpha), \quad N(-\alpha) = N(\alpha). \tag{93}$$

Let us limit our attention to trajectories lying in the region of phase space for which

$$N\int_0^1 \hat{\eta}_{\mathrm{H}}(N, tH, M)\,\mathrm{d}t - \hat{v}(N, H, M) < 0 \tag{94}$$

so that the right-hand side of (22) has the sign opposite to that of H:

$$\operatorname{sign}\{N\hat{\eta}(N, H, M) - H\hat{v}(N, H, M)\} = -\operatorname{sign} H. \tag{95}$$

This region certainly includes the half-space defined by $N \leqslant 0$. We define

$$\left\{\begin{matrix}\mathscr{H}^-(p)\\ \mathscr{H}^0(p)\\ \mathscr{H}^+(p)\end{matrix}\right\} \equiv \text{the set of all } (N, H, M) \text{ for which } N\hat{\mu}(N, H, M) + p\hat{v}(N, H, M)\left\{\begin{matrix}>\\ =\\ >\end{matrix}\right\}0. \tag{96}$$

$\mathscr{H}^-(p)$ is the region of phase space on which $H' < 0$, etc., by virtue of (21).

Now suppose that $(N(-\alpha), 0, 0) \in \mathscr{H}^-(p)$. Then the trajectory starting here immediately moves into the half-space where $H < 0$. It follows from (22) that M is increasing along the trajectory as long as $H < 0$ and (94) holds. If this trajectory does not enter into the region $\mathscr{H}^+$, then it cannot return to either the plane $H = 0$ or the plane $M = 0$, and therefore cannot possibly correspond to a solution of the boundary-value problem. If this trajectory does return to the plane $H = 0$, say at $s = -\alpha + \sigma$, then it must also return to the plane $M = 0$ at $s = -\alpha + 2\sigma$, as a consequence of the symmetry conditions. Such a trajectory is the upper trajectory sketched in Fig. 4. An analogous construction works when $(N(-\alpha), 0, 0) \in \mathscr{H}^+(p)$, and leads to the lower trajectory in Fig. 4. If $(N(-\alpha), 0, 0) \in \mathscr{H}^0(p)$, then this point is a critical point of (20)–(22), by definition of $\mathscr{H}^0$. We note that (20)–(22) admit the integral $N^2 + H^2 - 2pM = \text{const.}$, so that the trajectories in Fig. 4 lie on the paraboloid

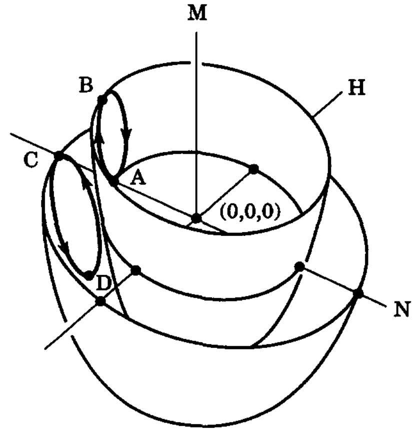

Fig. 4. Two trajectories ABA and CDC of (20)–(22) satisfying (93) for different values of p. The paraboloids on which they lie are given by (97). Points A and D lie in $\mathscr{H}^-$, and points B and C lie in $\mathscr{H}^+$.

$$M = \frac{1}{2p}[N^2 + H^2 - N(-\alpha)^2]. \quad (97)$$

If the material of the arch is hyperelastic, as is physically natural, we can get an alternative description of the trajectories. By definition of hyperelasticity there is a stored energy function W such that

$$\hat{N}(\nu,\eta,\mu) = \frac{\partial W}{\partial \nu}(\nu,\eta,\mu), \quad \hat{H}(\nu,\eta,\mu) = \frac{\partial W}{\partial \eta}(\nu,\eta,\mu), \quad \hat{M}(\nu,\eta,\mu) = \frac{\partial W}{\partial \mu}(\nu,\eta,\mu). \quad (98)$$

W is convex as a consequence of (13). Let $(N,H,M) \mapsto W^*(N,H,M)$ be the convex function conjugate to W (i.e. the Legendre transform of W), so that

$$\hat{\nu}(N,H,M) = \frac{\partial W^*}{\partial N}(N,H,M), \quad \hat{\eta}(N,H,M) = \frac{\partial W^*}{\partial H}(N,H,M),$$

$$\hat{\mu}(N,H,M) = \frac{\partial W^*}{\partial M}(N,H,M). \quad (99)$$

Then (20)–(22) admit the integral $W^*(N,H,M) = \text{const}$. This equation describes a nested family of unbounded convex surfaces in the phase space of Fig. 4. (The unboundedness is a consequence of the fact that the form of $\hat{\nu}(N,0,0)$ implies that its integral $W^*(N,0,0)$ is strictly increasing). Each trajectory in Fig. 4 lies on the intersection of surfaces of the form $W^*(N,H,M) = W^*(N(\alpha),0,0)$ and $N^2 + H^2 - 2pM = N(\alpha)^2$. These surfaces are tangent at $(N(\alpha),0,0)$ if there is a number λ such that $(W^*_N, W^*_H, W^*_M) = \lambda(N, H, -p)$ at this point, i.e. if $N(\alpha)\hat{\mu}(N(\alpha),0,0) + p\hat{\mu}(N(\alpha),0,0) = 0$. This is precisely the condition that $(N(\alpha), 0,0) \in \mathscr{H}^0$.

We now discuss conditions showing that trajectories like those of Fig. 4, which start in either $\mathscr{H}^-(p)$ or $\mathscr{H}^+(p)$, cannot generate solutions to the full boundary-value problem. Since the phase portrait Fig. 4 shows that a candidate trajectory must terminate on the plane $H = 0$, we conclude that $H(\pm\alpha) = 0$. Then (54) reduces to (56) into which we substitute (79) and use the symmetry to obtain

$$2\alpha\mu^{\#}\left(\frac{x(a)\sqrt{1+g'(x(\alpha))^2}\ \text{arccot}\,(-g'(x(\alpha)))}{\alpha}\right) = 2\theta(\alpha) \pmod{\pi}. \quad (100)$$

We limit our attention to configurations that evolve continuously from the undeformed circular state so we can drop the mod π from (100). We can supplement (100) with the following consequence of the force balance [*cf.* (58) and (62)]

$$\sin\theta(\alpha)N(\alpha) = -px(\alpha). \quad (101)$$

Since p and $x(\alpha)$ are positive, it follows that along any branch of solution pairs containing a trivial solution pair, neither $\sin\theta(\alpha)$ nor $N(\alpha)$ can vanish. Therefore we conclude that $\theta(\alpha) > 0$ and $N(\alpha) > 0$. We can replace $\theta(\alpha)$ and $x(\alpha)$ by their representations coming from (23) and (24):

$$\theta(\alpha) = \frac{1}{2}\int_{-\alpha}^{\alpha} \hat{\mu}(N(s),H(s),M(s))\,\mathrm{d}s,$$

$$x(\alpha) = \frac{1}{2}\int_{-\alpha}^{\alpha} [\hat{\nu}(N(s),H(s),M(s))\cos\theta(s) - \hat{\eta}(N(s),H(s),M(s))\sin\theta(s)]\,\mathrm{d}s. \quad (102)$$

For a given set of constitutive functions there are regions of (N,H,M,p)-space with the property that no nontrivial solution pair lying in such a region can satisfy (100)–(102). Thus we conclude that in these regions the nodal properties of H are preserved.

Let us indicate how such regions can be found. In the formulation of Antman and Dunn (1980), which we are rejecting, $\mu^{\#} = 1$. Thus (100), (102) reduce to

$$2\alpha = \int_{-\alpha}^{\alpha} \hat{\mu}(N(s), H(s), M(s))\, ds. \tag{103}$$

In the regions introduced in (95) it follows that every trajectory lies either entirely above or entirely below the plane $M = 0$, which is the locus where $\hat{\mu}(N, H, M) = 1$. Thus on such trajectories $\hat{\mu}(N(s), H(s), M(s)) - 1$ is either everywhere positive or everywhere negative except for a finite number of points. Thus (103) cannot hold on such trajectories.

If $\mu^{\#}(v) = v$, so that (80) holds, then (100)–(102) yield

$$2 \arcsin x(\alpha) = 2\theta(\alpha) = \int_{-\alpha}^{\alpha} \hat{\mu}(N(s), H(s), M(s))\, ds, \quad N(\alpha) = -p. \tag{104a,b}$$

It follows from (80) and the first equality of (104a) that $0 < \theta(\alpha) < \alpha$. One could therefore show that (104a) cannot hold by showing that enough of the trajectory lies either in the region where $\hat{\mu}(N, H, M) > 1$ or in the region where $\hat{\mu}(N, H, M) < 0$ for the integral in (104a) to lie outside the range $[0, \alpha]$. For this purpose it might be helpful to use the integrals $2pM = N^2 + H^2 - p^2$ and $W^*(N, H, M) = W^*(-p, 0, 0)$. We pursue this question neither for this special case nor for the general case of (100)–(102) because its resolution would no doubt require estimates on the constitutive functions so technical that the physical significance of the results would be obscured. Instead, we carry out a detailed local analysis of the problem for the doubly-hinged arch in the next section.

7. POST-BUCKLING BEHAVIOR FOR SIMPLE EIGENVALUES

We now carry out a perturbation analysis [*cf.* Keller (1968)] of the solutions in the neighborhood of a bifurcation point from the trivial branch when $p = p_0$ is a simple eigenvalue of the boundary-value problem linearized about the trivial state.

Let ε be a small amplitude parameter to be identified below. We seek solution pairs of our boundary-value problems in the form

$$p(\varepsilon) = p_0 + \varepsilon p_1 + \frac{\varepsilon^2}{2} p_2 + o(\varepsilon^2),$$
$$H(s, \varepsilon) = \varepsilon H_1(s) + \frac{\varepsilon^2}{2} H_2(s) + \frac{\varepsilon^3}{3!} H_3(s) + o(\varepsilon^3), \text{ etc.} \tag{105}$$

If the constitutive functions are sufficiently smooth, then a version of the Implicit Function Theorem ensures that nontrivial solution branches have this character near a bifurcation point at a simple eigenvalue [*cf.* Crandall and Rabinowitz (1971)]. Since $H_k(s) = \partial^k H(s, \varepsilon)/\partial\varepsilon^k|_{\varepsilon=0}$, etc., we can find the equations satisfied by the kth perturbation by substituting (105) into the boundary-value problem, differentiating its equations k times with respect to ε, and then setting $\varepsilon = 0$.

We let $(N_0(p), M_0(p))$ denote the solution of one of the trivial problems treated in Section 5. For any constitutive function such as $\bar{v}_N$ we set $v_N^0(p_0) \equiv \hat{v}_N(N_0(p_0), 0, M_0(p_0))$, etc., and usually suppress the argument p_0. We introduce the notation

$$\beta \equiv \mu^0 + p_0 \eta_H^0, \tag{106a}$$

$$\sigma \equiv \mu^0 v_N^0 - v^0 \mu_N^0, \quad \tau \equiv \mu^0 v_N^0 - v^0 v_M^0, \tag{106b,c}$$

$$\chi \equiv v^0 \mu_M^0 - \mu^0 v_M^0, \quad \omega \equiv v^0 \mu_M^0 - \mu^0 \mu_N^0, \tag{106d,e}$$

$$\kappa \equiv \mu^0\sigma+\nu^0\chi \equiv \mu^0\tau+\nu^0\omega \equiv (\mu^0)^2\nu_N^0-\nu^0\mu^0(\mu_N^0+\nu_M^0)+(\nu^0)^2\mu_M^0 > 0. \tag{106f}$$

κ is positive by (13). We make the mild assumption that each of these constants is positive. (For hyperelastic materials $\sigma = \tau$ and $\chi = \omega$.) Let $\mathbf{u}_k \equiv (N_k, H_k, M_k, \theta_k, x_k, y_k)$. In matrix equations, $\mathbf{u}_k$ is represented as a column vector. Using (66), we readily find that $\mathbf{u}_k$ satisfies a system of the following form:

$$\mathbf{u}_k' - \mathbf{Q}\mathbf{u}_k = p_k\mathbf{e}+\mathbf{f}_k, \tag{107a}$$

where

$$\mathbf{Q} \equiv \begin{bmatrix} 0 & \beta & 0 & 0 & 0 & 0 \\ -\mu^0-\dfrac{p_0}{\mu^0}\sigma & 0 & \dfrac{p_0}{\mu_0}\chi & 0 & 0 & 0 \\ 0 & -\dfrac{\nu^0\beta}{\mu^0} & 0 & 0 & 0 & 0 \\ \mu_N^0 & 0 & \mu_M^0 & 0 & 0 & 0 \\ \nu_N^0\cos\mu^0 s & -\eta_H^0\sin\mu^0 s & \nu_M^0\cos\mu^0 s & -\nu^0\sin\mu^0 s & 0 & 0 \\ \nu_N^0\sin\mu^0 s & \eta_H^0\cos\mu^0 s & \nu_M^0\sin\mu^0 s & \nu^0\cos\mu^0 s & 0 & 0 \end{bmatrix}, \tag{107b}$$

$$\mathbf{e} \equiv (0, -\nu^0, 0, 0, 0, 0), \tag{107c}$$

and $\mathbf{f}_k$ depends on the first four components of each of $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ and on $p_0, \ldots, p_{k-1}$. In particular, $\mathbf{f}_1 = \mathbf{0}$, and $\mathbf{f}_2 = (f_{21}, \ldots, f_{26})$ with

$$\begin{aligned}
f_{21} &= [p_1\eta_H^0+2(\mu_N^0N_1+\mu_M^0M_1)+2p_0(\eta_{NH}^0N_1+\eta_{MH}^0M_1)]H_1,\\
f_{22} &= -2p_1(\nu_N^0N_1+\nu_M^0M_1)-2(\mu_N^0N_1+\mu_M^0M_1)N_1-p_0\left[\left(\nu_{NN}^0-\frac{\nu^0}{\mu^0}\mu_{NN}^0\right)N_1^2\right.\\
&\quad \left.+\left(\nu_{HH}^0-\frac{\nu^0}{\mu^0}\mu_{HH}^0\right)H_1^2+\left(\nu_{MM}^0-\frac{\nu^0}{\mu^0}\mu_{MM}^0\right)M_1^2+2\left(\nu_{NM}^0-\frac{\nu^0}{\mu^0}\mu_{NM}^0\right)N_1M_1\right],\\
f_{23} &= -2[\nu_N^0N_1+\nu_M^0M_1-\eta_H^0N_1+\eta_{NH}^0N_1+\eta_{MN}^0M_1]H_1,\\
f_{24} &= \mu_{NN}^0N_1^2+\mu_{HH}^0H_1^2+\mu_{MM}^0M_1^2+2\mu_{NM}^0N_1M_1,\\
f_{25} &= (\nu_{NN}^0N_1^2+\nu_{HH}^0H_1^2+\nu_{MM}^0M_1^2+2\nu_{NM}^0N_1M_1)\cos\mu^0 s-2(\nu_N^0N_1+\nu_M^0M_1)\theta_1\sin\mu^0 s\\
&\quad -\nu^0\theta_1^2\cos\mu^0 s-2\eta_H^0\theta_1H_1\cos\mu^0 s-2(\eta_{NH}^0N_1+\eta_{MH}^0M_1)H_1\sin\mu^0 s,\\
f_{26} &= -(\nu_{NN}^0N_1^2+\nu_{HH}^0H_1^2+\nu_{MM}^0M_1^2+2\nu_{NM}^0N_1M_1)\sin\mu^0 s-2(\nu_N^0N_1+\nu_M^0M_1)\theta_1\cos\mu^0 s\\
&\quad +\nu^0\theta_1^2\sin\mu^0 s+2\eta_H^0\theta_1H_1\sin\mu^0 s-2(\eta_{NH}^0N_1+\eta_{MH}^0M_1)H_1\cos\mu^0 s.
\end{aligned} \tag{108}$$

From these equations or from (25) and (26) we obtain

$$H_k''+q(p_0)^2H_k = \mathscr{H}_k, \tag{109}$$

$$\theta_k'' = -\beta\frac{\omega}{\mu^0}H_k+\Theta_k, \tag{110}$$

where

$$q^2 \equiv \frac{\beta}{\mu^0}\left[(\mu^0)^2+p_0\left(\frac{\nu^0}{\mu_0}\chi+\sigma\right)\right] \equiv \beta\left[\mu^0+\frac{p_0\kappa}{(\mu^0)^2}\right], \tag{111}$$

and $\mathscr{H}_k$ and Θ_k depend on the first three components of each of $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ and on

$p_0, \ldots, p_{k-1}$. In particular, $\mathscr{H}_1 = 0 = \Theta_1$. Since η_H^0 and κ are positive by (13) and since we have shown that μ^0 must be positive, it follows that

$$q^2 > (\mu^0)^2 \quad \text{for } p_0 > 0, \quad \frac{\partial q^2}{\partial p_0}(0) > 0. \tag{112}$$

The general solution of (107) for $k = 1$ includes

$$H_1(s) = A \cos qs + B \sin qs, \tag{113a}$$

$$N_1(s) = \frac{\beta}{q}(A \sin qs - B \cos qs + C), \tag{113b}$$

$$M_1(s) = -\frac{\beta v^0}{\mu^0 q}(A \sin qs - B \cos qs + D), \tag{113c}$$

$$\theta_1(s) = \frac{\beta\omega}{\mu^0 q^2}(A \cos qs + B \sin qs + Es + F), \tag{113d}$$

where $A, \ldots, F$ are constants of integration satisfying

$$[(\mu^0)^2 + p_0\sigma]C + \frac{p_0\chi v^0}{\mu^0}D = -p_1\frac{v^0\mu^0 q}{\beta}, \tag{113e}$$

$$-\mu^0\mu_N^0 C + v^0\mu_M^0 D = -\frac{\omega}{q}E. \tag{113f}$$

Notice the substantial algebraic simplifications that would arise in the preceding development if the material were assumed to be inextensible and unshearable.

Doubly-welded arch

For the doubly-welded arch whose ends move in radial grooves, we found that $\mu^0 = 1$. The first perturbation is subject to the following boundary conditions, obtained from (56):

$$H_1(\pm\alpha) = 0, \quad \theta_1(\pm\alpha) = 0. \tag{114}$$

Substituting (113) into (114) we obtain asymmetric nontrivial solutions defined by

$$B = 0, \quad E = 0, \quad F = 0 \quad \text{when} \quad q(p_0) = \frac{(2j+1)\pi}{2\alpha}, \tag{115}$$

where j is an integer. We likewise obtain symmetric nontrivial solutions defined by

$$A = 0, \quad E = 0, \quad F = 0 \quad \text{when} \quad q(p_0) = \frac{l\pi}{\alpha}, \tag{116}$$

where l is a nonzero integer. The substitution of (115) or (116) into (113e,f) reduces it to a linear system for C and D. Condition (13) ensures that the matrix of coefficients of C and D in this system is negative for $p_0 \geqslant 0$. Thus C and D are each proportional to p_1, so that there are numbers Γ and Δ, depending on the first derivatives of the constitutive functions at the trivial solution, such that

$$C = \Gamma p_1, \quad D = \Delta p_1. \tag{117}$$

Note that the representations for H_1 and θ_1 from (115) and (116) are independent of p_1. It is typical of our perturbation methods that p_1 is not determined by the first perturbation.

Suppose that p is slowly raised from 0. If $\alpha > \pi/2$, then (115) with $j = 0$ gives a value of $q < 1$. It can never be attained; thus there is no eigenfunction for it. Consequently the first eigenvalue that p could reach is that corresponding to (16) with $l = 1$, for which $q = \pi/\alpha > 1$. Thus the lowest buckling load (if it exists) corresponds to an arch symmetrically deformed about the vertical. On the other hand, if $\alpha < \pi/2$, then the lowest buckling load (if it exists) corresponds to an asymmetrically deformed arch. If $\alpha = \pi/2$, the arch is semicircular. Is $p = 0$ an eigenvalue corresponding to (115) with $j = 0$? To resolve this question we compute the first perturbation $\mathbf{r}_1$ of $\mathbf{r}$. We find that it cannot satisfy the linearized version of (90) unless $A_1 = 0$, in which case the solution is trivial. Thus if $\alpha = \pi/2$, the lowest buckling load (if it exists) corresponds to (116) with $l = 1$, and the lowest buckling mode is symmetric.

At this stage it is convenient to define the parameter ε by

$$\varepsilon \equiv \begin{cases} \dfrac{1}{\sqrt{\alpha}} \displaystyle\int_{-\alpha}^{\alpha} H(s,\varepsilon)\cos qs \, \mathrm{d}s & \text{if} \quad q(p_0) = \dfrac{(2k+1)\pi}{2\alpha}, \\ \dfrac{1}{\sqrt{\alpha}} \displaystyle\int_{-\alpha}^{\alpha} H(s,e)\sin qs \, \mathrm{d}s & \text{if} \quad q(p_0) = \dfrac{l\pi}{\alpha}. \end{cases} \tag{118}$$

Differentiating (118) with respect to ε, setting $\varepsilon = 0$, and using (113), (115) and (116) we obtain $A = 1/\sqrt{\alpha} = B$. (The use of (118) greatly simplifies the formulae for the higher perturbations without loss of generality.)

Let us now turn to the second perturbation, governed by (109) with $k = 2$. Here $\mathscr{H}_2$ has the form

$$\mathscr{H}_2 = 2\frac{p_0}{\mu_0}[\nu^0\mu^0_{\mathrm{HH}} - \mu^0\nu^0_{\mathrm{HH}}]H_1H_1' + [\Phi N_1 + \Psi M_1 + \Omega p_1]H_1, \tag{119}$$

where Φ and Ψ are complicated expressions involving the first and second derivatives of the constitutive functions at the trival solution, and where Ω is a complicated expression involving just the first derivatives of the constitutive functions at the trivial solution.

H_2 satisfies the boundary conditions $H_2(\pm\alpha) = 0$, which come from (56). The Alternative Theorem asserts that (109) for $k = 2$ has a solution satisfying these boundary conditions if and only if its right-hand side (119) is orthogonal to the solution H_1 of the homogeneous problem in the sense that the integral over $(-\alpha, \alpha)$ of the product of H_1 with (119) vanishes. From (113), (115), (116) we readily see that the only terms that survive this integration process are those containing the constants C, D, p_1. Thus the orthogonality condition reduces to

$$(\Phi\Gamma + \Psi\Delta + \Omega)p_1 = 0. \tag{120}$$

It can be shown that the term in parentheses vanishes if and only if $(\mathrm{d}q^2/\mathrm{d}p_0) = 0$, i.e. if and only if the eigenvalue p_0 is not algebraically simple. (It is clear from the dependence of Φ and Ψ on the second derivatives of constitutive functions, which are not subject to any local restrictions, that the term in parentheses in (120) is generically not zero. We briefly discuss the case of a double eigenvalue in the next section.) It follows that if p_0 is simple, then $p_1 = 0$. This means that neither the symmetric nor asymmetric buckled states are generated by a transcritical bifurcation, with its attendant implications for instability. This fact is not surprising for the asymmetric modes, each configuration of which has an image reflected across the y-axis. But this fact is not obvious for the symmetric modes, for which it implies that the first-order terms given by (113), (116) have an up–down symmetry, which cannot be expected to be preserved in the higher-order corrections.

Using the condition that $p_1 = 0$, which simplifies the formulae for N_1 and M_1 and thus simplifies the problem for H_2, we can solve for H_2 and the other variables of the second perturbation. The resulting formula is simplified by the requirement that H_2 be orthogonal to H_1, which comes from (118). Applying the Alternative Theorem to the third perturbation shows that generically $p_2 \neq 0$, so that the bifurcation pattern is that of a pitchfork. The sign of p_2, which determines important stability questions (because $p_1 = 0$), is dictated by conditions involving derivatives up to order three of the constitutive functions, i.e. dictated by the nonlinear constitutive response. We do not carry out the details of the requisite computations, since they are standard.

Doubly hinged arch with horizontal ends

We now treat the far more difficult problem with boundary conditions (53)–(55), (57), (65), which yield

$$y_1(\pm\alpha) = \pm g'(x_0(\alpha))x_1(\pm\alpha), \quad y_1(\alpha) = y_1(-\alpha), \tag{121a,b}$$

$$M_1(\pm\alpha) = 0. \tag{121c}$$

$$\begin{aligned} H_1(\alpha) - H_1(-\alpha) - p_0\frac{v_0}{\mu^0}[\theta_1(\alpha) - \theta_1(-\alpha)] &= -2p_0\frac{v^0}{\mu^0}g''(x_0(\alpha))x_1(\alpha)\sin^2\mu^0\alpha \\ &= -\frac{2p_0 v^0\mu^0\alpha\mu_{\mathrm{N}}^0}{\mu^0 v_{\mathrm{N}}^0\tan\mu^0\alpha + v^0\mu_{\mathrm{N}}^0(\sin\mu^0\alpha - \mu^0\alpha\cos\mu^0\alpha)}. \end{aligned} \tag{121d}$$

[The second equality of (121d) comes from the analog of (85) via (72).]

Applying the boundary conditions (121c) to (113c) we obtain

$$A\sin q\alpha = 0, \quad D = B\cos q\alpha. \tag{122a,b}$$

Substituting (113) into the last row of (107a) and using (121b) we obtain

$$\begin{aligned} 0 = y_1(\alpha) - y_1(-\alpha) &= \int_{-\alpha}^{\alpha} y_1'\,\mathrm{d}s \\ &= \frac{2v^0\beta\omega}{(\mu^0 q)^2}F\sin\mu^0\alpha + A\left\{\frac{\beta\tau}{\mu^0 q}\left[\frac{\sin(q-\mu^0)\alpha}{q-\mu^0} - \frac{\sin(q+\mu^0)\alpha}{q+\mu^0}\right]\right. \\ &\quad \left. + \left(\eta_{\mathrm{H}}^0 + \frac{v^0\beta\omega}{\mu^0 q^2}\right)\left[\frac{\sin(q-\mu^0)\alpha}{q-\mu^0} + \frac{\sin(q+\mu^0)\alpha}{q+\mu^0}\right]\right\}. \end{aligned} \tag{123}$$

Substituting (113) into (121d), we obtain

$$-p_0\frac{v^0\beta\omega}{(\mu^0 q)^2}E\alpha + B\left[1 - p_0\frac{v^0\beta\omega}{(\mu^0 q)^2}\right]\sin q\alpha = -p_0\frac{v^0}{\mu^0}g''(x_0(\alpha))x_1(\alpha)\sin^2\mu^0\alpha \tag{124a}$$

where

$$\begin{aligned} x_1(\alpha) = &-\frac{B}{2}\left\{\frac{\beta\tau}{\mu^0 q}\left[\frac{\sin(q-\mu^0)\alpha}{q-\mu^0} + \frac{\sin(q+\mu^0)\alpha}{q+\mu^0}\right]\right. \\ &\left. + \left(\eta_{\mathrm{H}}^0 + \frac{v^0\beta\omega}{\mu^0 q^2}\right)\left[\frac{\sin(q-\mu^0)\alpha}{q-\mu^0} - \frac{\sin(q+\mu^0)\alpha}{q+\mu^0}\right]\right\} \\ &+ \frac{\beta}{(\mu^0)^2 q}(\mu^0 v_{\mathrm{N}}^0 C - v^0 v_{\mathrm{M}}^0 D)\sin\mu^0\alpha + \frac{Ev^0\beta\omega}{(\mu^0 q)^2}(\alpha\mu^0\cos\mu^0\alpha - \sin\mu^0\alpha) \end{aligned} \tag{124b}$$

by (107) and (113).

We thus have the system (113e,f), (122)–(124) for the constants $A, \ldots, F$. This system uncouples into (122a) and (123) for A and F, and the system (113ef), (122b), (124) for B, C, D, E, which we write as

$$\mathbf{R}\begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix} \equiv \begin{bmatrix} 0 & (\mu^0)^2 + p_0\sigma & p_0\chi v^0/\mu^0 & 0 \\ 0 & -\mu^0\mu_N^0 & v^0\mu_M^0 & \omega/q \\ \cos q\alpha & 0 & -1 & 0 \\ b & c & d & e \end{bmatrix}\begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} -p_1 v^0\mu^0 q/\beta \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{125}$$

where the components b, c, d, e are found from (124). The solvability of our boundary-value problem is equivalent to the solvability of (122), (123), (125). To determine the value of p_1 we compute the second perturbation and then invoke the Alternative Theorem, which says that the nonhomogeneous linear equations of the second perturbation have a solution if and only if the nonhomogeneous term is orthogonal to the null space of the adjoint problem. The computations necessary to apply this theorem are far more extensive than those used for the doubly-welded arch because here our problem does not reduce to one essentially governed by a single second-order equation, namely (109), (118). Thus we must confront the full system. We first construct the adjoint and then we return to the analysis of (125) and the second perturbation.

Let $\mathbf{u} \equiv (N, H, M, \theta, x, y)$ and $\mathbf{u}_* \equiv (N_*, H_*, M_*, \theta_*, x_*, y_*)$. We define the inner product of these two sextuples to be

$$\langle \mathbf{u}, \mathbf{u}_* \rangle \equiv \int_{-\alpha}^{\alpha} (NN_* + HH_* + MM_* + \theta\theta_* + xx_* + yy_*)\, \mathrm{d}s. \tag{126}$$

The adjoint $\mathbf{L}^*$ of the differential operator $\mathbf{Lu} \equiv \mathbf{u}' - \mathbf{Qu}$ defined on those $\mathbf{u}$ satisfying (121) satisfies $\langle \mathbf{Lu}, \mathbf{u}_* \rangle = \langle \mathbf{u}, \mathbf{L}^*\mathbf{u}_* \rangle$ [*cf.* Naimark (1967) for precise definitions]. We find that $\mathbf{L}^*\mathbf{u}_* = -\mathbf{u}_*' - \mathbf{Q}^*\mathbf{u}_*$ where $\mathbf{Q}^*$ is the transpose of $\mathbf{Q}$. The domain of $\mathbf{L}^*$ consists of those $\mathbf{u}_*$ satisfying the boundary conditions

$$N_*(\alpha) = 0 = N_*(-\alpha), \quad H_*(\alpha) = H_*(-\alpha), \quad p_0 v^0 H_*(\pm\alpha) + \mu^0\theta_*(\pm\alpha) = 0,$$

$$2\frac{p_0\mu^0}{v^0} g''(x_0(\alpha))H_*(\alpha)\sin^2\mu^0\alpha - x_*(\alpha) - x_*(-\alpha) + [y_*(\alpha) - y_*(-\alpha)]\cot\mu^0 = 0. \tag{127}$$

The Alternative Theorem says that eqns (107) for $k = 1, 2$ have solutions if and only if

$$\langle p_1\mathbf{e}, \mathbf{u}_* \rangle \equiv -v^0 p_1 \int_{-\alpha}^{\alpha} H_*(s)\, \mathrm{d}s = 0, \tag{128a}$$

$$\langle p_2\mathbf{e} + \mathbf{f}_2, \mathbf{u}_* \rangle \equiv -v^0 p_2 \int_{-\alpha}^{\alpha} H_*(s)\, \mathrm{d}s + \langle \mathbf{f}_2, \mathbf{u}_* \rangle = 0, \tag{128b}$$

respectively for all $\mathbf{u}_*$ satisfying the homogeneous adjoint system $\mathbf{u}_*' + \mathbf{Q}^*\mathbf{u}_* = \mathbf{0}$ and the boundary conditions (127). We find that the general solution of this homogeneous adjoint system is

$$x_* = \text{const.}, \quad y_* = \text{const.},$$

$$\theta_*(s) = -\frac{v^0}{\mu^0}(x_* \cos \mu^0 s + y_* \sin \mu^0 s) + E_*,$$

$$H_*(s) = A_* \cos qs + B_* \sin qs - \frac{\beta\omega}{\mu^0 q^2} E_* + \frac{\beta\kappa(x_* \cos \mu^0 s + y_* \sin \mu^0 s)}{(\mu^0)^2[q^2 - (\mu^0)^2]},$$

$$N_*(s) = \left(\mu^0 + \frac{p_0}{\mu^0}\sigma\right) \int H_* - \mu_N^0 \int \theta_* - \frac{v_N^0}{\mu^0}(x_* \sin \mu^0 s - y_* \cos \mu^0 s) + C_*,$$

$$M_*(s) = -\frac{p_0}{\mu^0}\chi \int H_* - \mu_M^0 \int \theta_* - \frac{v_M^0}{\mu^0}(x_* \sin \mu^0 s - y_* \cos \mu^0 s) + D_*, \tag{129a}$$

with

$$\int H_* \equiv \frac{A_*}{q} \sin qs - \frac{B_*}{q} \cos qs - \frac{\beta\omega}{\mu^0 q^2} E_* s + \frac{\beta\kappa(x_* \sin \mu^0 s - y_* \cos \mu^0 s)}{(\mu^0)^3[q^2 - (\mu^0)^2]},$$

$$\int \theta_* \equiv -\frac{v^0}{(\mu^0)^2}(x_* \sin \mu^0 s - y_* \cos \mu^0 s) + E_* s, \tag{129b}$$

where $A_*, \ldots, E_*$ are constants of integration satisfying

$$G y_* + \mu^0 C_* - v^0 D_* = 0 \tag{129c}$$

with G a computable constant. The boundary conditions (127) imply that

$$y_* = 0, \quad B_* \sin q\alpha = 0, \quad B_* \cos q\alpha + qC_* = 0, \quad \mu^0 C_* - v^0 D_* = 0, \tag{130}$$

$$\frac{p_0 v^0}{\mu^0}\left[A_* \cos q\alpha - \frac{\beta\omega}{\mu^0 q^2} E_* + \frac{\beta\kappa x_* \cos \mu^0\alpha}{(\mu^0)^2[q^2 - (\mu^0)^2]}\right] + E_* - \frac{v^0}{\mu^0} x_* \cos \mu^0\alpha = 0, \tag{131a}$$

$$x_* + \left(E_* - \frac{v^0}{\mu^0} x_* \cos \mu^0\alpha\right) g''(x_0(\alpha)) \sin^2 \mu^0\alpha = 0, \tag{131b}$$

$$-[(\mu^0)^2 + p_0\sigma]\left[\frac{A_*}{q} \sin q\alpha - \frac{\beta\omega}{\mu^0 q^2} E_*\alpha + \frac{\beta\kappa}{q^2 - (\mu^0)^2} \frac{x_* \sin \mu^0\alpha}{(\mu^0)^3}\right]$$
$$+ \mu_N^0 E_*\alpha + \frac{\sigma}{(\mu^0)^2} x_* \sin \mu^0\alpha = 0. \tag{131c}$$

Now we are ready to analyse the first and second perturbations. Condition (122a) implies that either $A = 0$ or $\sin q\alpha = 0$. We break up the problem into a few cases:

Case (i): $\sin q\alpha = 0$, $A \neq 0$. Since p_0 is restricted by the characteristic equation $\sin q\alpha = 0$, we restrict our study to the generic case that $\det \mathbf{R} \neq 0$. Then (125) implies that B, C, D, E are each proportional to p_1. We likewise make the generic assumption that the determinant of coefficients of (131) is not zero. Then $A_* = E_* = x_* = 0$, and (129) yields $H_*(s) = B_* \sin qs$. Substituting this, (108), and our special form of (113) into (128) we obtain an equation of the form $p_1 AG = 0$ where G is a complicated function of the first and second derivatives of the constitutive functions evaluated at the trivial solution. In the generic case that $G \neq 0$, we find that $p_1 = 0$. Thus by the arguments given above for the doubly-welded arch, the bifurcation diagram generically has the form of a pitchfork. This is to be expected because the solutions, which are asymmetrical with respect to the y-axis, come in pairs, which are reflections of each other in this axis.

Case (ii) : $A = 0$, $\sin q\alpha \neq 0$, $\det \mathbf{R} \neq 0$. This last condition again implies that B, C, D, E are each proportional to p_1. Then substituting (108) into (128) we obtain equations of the form

$$p_1 \int_{-\alpha}^{\alpha} H_*(s)\,ds = 0, \quad -v^0 p_2 \int_{-\alpha}^{\alpha} H_*(s)\,ds + p_1^2 G = 0, \tag{132a, b}$$

where G is a (different) complicated function of the first and second derivatives of the constitutive functions evaluated at the trivial solution. If $\int_{-\alpha}^{\alpha} H_*(s)\,ds \neq 0$, then (132) implies that $p_1 = 0$; if $\int_{-\alpha}^{\alpha} H_*(s)\,ds = 0$, then (132) implies that $p_1^2 G = 0$. In the generic case that $G \neq 0$, we again find that $p_1 = 0$. Thus, generically, $p_1 = 0$, and the solution of the first perturbation is trivial.

Case (iii) : $A = 0$, $\sin q\alpha \neq 0$, $\det \mathbf{R} = 0$. In this case the homogeneous problem

$$\mathbf{R}^* \begin{pmatrix} B^+ \\ C^+ \\ D^+ \\ E^+ \end{pmatrix} = \mathbf{0}$$

adjoint to (125) has a nontrivial solution. If there is a nontrivial solution with $B^+ = 0$, then four 3×3 subdeterminants of $\mathbf{R}^*$ must each vanish, which is nongeneric. Otherwise, the Alternative Theorem applied to (125) requires that $p_1 B^+ = 0$ and hence $p_1 = 0$. [An alternative proof can be based on (128).] But here the solution of the first perturbation is not trivial. It corresponds to a solution symmetrical about the y-axis. Just as for the arch with welded ends, the bifurcation has a generic pitchfork form. The characteristic equation for the critical pressures is $\det \mathbf{R} = 0$.

Case (iv) : $A = 0$, $\sin q\alpha = 0$. If $\det \mathbf{R} \neq 0$, then just as in case (ii) we find that the solution is trivial. If $\det \mathbf{R} = 0$, then it is not generic for this condition and $\sin q\alpha = 0$ to occur simultaneously.

The result that the bifurcation of symmeric solutions is generically described by a pitchfork is rather surprising: Consider the degenerate case of this problem obtained by assuming that the arch is inextensible and unshearable, so that $\hat{v} = 1$ and $\hat{\eta} = 0$. In this case, the trivial solutions correspond to rigid displacements of the reference configuration. Accordingly we take the boundary conditions for the nonlinear problem to be (86). An analysis similar to that just outlined shows that there are both symmetric and asymmetric buckled states and that the symmetric ones are reached by a transcritical bifurcation.

8. BIFURCATION FROM MULTIPLE EIGENVALUES. NONBIFURCATING BRANCHES

Recall that the eigenvalues of the linearized problem depend on the form of q^2. In particular, for the doubly-welded arch, the eigenvalues are those values of p_0 at which the graph of q intersects certain horizontal lines [*cf.* (115), (116)]. Now for the inextensible, unshearable arch discussed at the end of Section 7, we find that $q^2 = 1 + \mu_M^0 p_0$, i.e. q^2 increases linearly with p_0. For extensible, unshearable arches with constitutive functions constructed as in Section 3, however, the penalization of total compression in the three-dimensional equations often leads to qs that are bounded and whose height decreases pointwise as a thickness parameter h increases [*cf.* Antman and Pierce (1990)]. For shearable rods, the presence of $\eta_H^0 p_0$ in the factor β in (111) can (but need not) markedly change the behavior of q^2 for large p_0. For the material defined by (52), the crucial asymptotic behavior of $\eta_H^0(p_0)$ would be determined by the term containing the parameter D.

Suppose that the graph of q has a single point (a local maximum) with slope zero. This implies that there are just a finite number of eigenvalues. Then as h is increased and the height of q is consequently decreased, various eigenvalues coalesce and disappear. (Of course, we cannot vary h for a given arch.) These eigenvalues have multiplicity 2 at their coalescence. In physical terms the arch is made too stiff to admit most and possibly all buckled states near the trivial state. On the other hand, the arch may certainly admit large nontrivial states that do not arise from the standard bifurcation process. We can determine their properties by studying the bifurcation problem parametrized by both p and h and using generalizations of the theorems supporting the treatment of Section 6. A key step in such an analysis is the determination of the local behavior near a double eigenvalue. Since the details of such a study, which combines the Poincaré shooting technique with the singularity theory of Golubitsky and Schaeffer (1985), are just complicated versions of those employed by Antman and Pierce (1990), and Antman and Marlow (1992), we omit them. The explicit examples of constitutive equations described at the end of Section 3 are very useful in illuminating these issues [cf. Antman and Pierce (1990)].

9. COMMENTS

Our refined constitutive equations were inspired from those of the three-dimensional theory. They account for couplings between effects of bending and stretching, which are far from obvious in a purely one-dimensional theory, and for effects due to the thickness of the arch, which depend on the precise material surface to which the pressure is applied. Many of the essential novelities of our model are embodied in Fig. 2.

The main consequences of these refined constitutive equations are the need to solve an inverse problem to determine groove shapes that admit trivial solutions and the absence of intuitive criteria to determine whether symmetric solutions are generated by transcritical bifurcations. Many of the methods used by Antman and Dunn (1980) in their study of the buckling of nonlinearity elastic arches (with what we might in retrospect call their naive constitutive theory) had to be significantly altered to handle the present theory. (Their work should be consulted for earlier references.) In particular, their global bifurcation theory for many problems other than the doubly-welded arch loses much of its simplicity when our constitutive assumptions are used; see the development in Section 6 beginning with the paragraph containing (92). We also note that our results have a character quite different from those of Antman and Marlow (1992) for the buckling of columns under end thrust.

We found that the symmetric buckled states of doubly-welded arches and doubly-hinged arches whose ends slide in grooves are generated by pitchfork bifurcations. On the other hand, the symmetric buckled states of inextensible doubly-hinged arches with fixed ends are generated by transcritical bifurcations. Since the symmetric buckled states do not come in mirror-image pairs, the bifurcation diagram is not symmetric about the trivial branch. That this lack of symmetry is not manifested in the first perturbation, i.e. that there is no transcritical bifurcation, for the problems treated in Section 7 may be attributed to the presence of the grooves: Their effect may be to reduce the difference in stiffness between "upward" and "downward" buckling.

The role of the shear stress H dominated our analysis. It defines a trivial solution and its nodal properties globally characterize branches. One could also study nodal properties of other functions, but their complexity makes the effort unrewarding.

Shearability enters our analysis in several ways: At the beginning of Section 8 we point out how it can markedly change the disposition of eigenvalues from that for an unshearable arch. The presence of shearability also influences the local post-buckling behavior, as is evident from the simplifications in (108) that would attend the assumptions that $\hat{\eta} = 0$ and that the other constitutive functions be independent of H. Shearability also cuts down the region in phase space in which various qualitative results can hold, e.g. (94), which is a significant restriction, reduces to a triviality for an unshearable rod. Moreover, for unshearable arches, the integrals discussed at the end of Section 6 become much more useful than they are here.

As might be expected, the difficulties caused by our refined constitutive equations are magnified for dynamical problems. The specific choice of coordinates that leads to the attractive (41) results in compensatory troubles in the inertial terms (simply because for circular arches or rings in which $x^1 = 0$ defines the curve of centroids, the mass of the material with $x^1 < 0$ exceeds the mass of the material with $x^1 > 0$).

One may ask why we should bother analysing unstable buckled states. The answer, confirmed in extensive recent studies, is they are responsible for the richness of the dynamical response. In fact, in many motions of many discrete systems, such as the rotating motion of a pendulum, the configuration spends most of its time in a neighborhood of unstable equilibrium states.

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